

UNITAL q -POSITIVE MAPS ON $M_2(\mathbb{C})$ AND A RELATED E_0 -SEMIGROUP RESULT

CHRISTOPHER JANKOWSKI

ABSTRACT. From previous work, we know how to obtain type II₀ E_0 -semigroups using boundary weight doubles (ϕ, ν) , where $\phi : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$ is a unital q -positive map and ν is a normalized unbounded boundary weight over $L^2(0, \infty)$. In this paper, we classify the unital q -positive maps $\phi : M_2(\mathbb{C}) \rightarrow M_2(\mathbb{C})$. We find that every unital q -pure map $\phi : M_2(\mathbb{C}) \rightarrow M_2(\mathbb{C})$ is either rank one or invertible. We also examine the case $n = 3$, finding the limit maps L_ϕ for all unital q -positive maps $\phi : M_3(\mathbb{C}) \rightarrow M_3(\mathbb{C})$. In conclusion, we present a cocycle conjugacy result for E_0 -semigroups induced by boundary weight doubles (ϕ, ν) when ν has the form $\nu(\sqrt{I - \Lambda(1)}B\sqrt{I - \Lambda(1)}) = (f, Bf)$.

1. INTRODUCTION

A linear map $\phi : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$ with no negative eigenvalues is said to be q -positive if $\phi(I + t\phi)^{-1}$ is completely positive for all $t \geq 0$. This class of maps has recently played a key role in constructing E_0 -semigroups in [7]. Let H be a separable Hilbert space whose inner product (\cdot, \cdot) is conjugate-linear in its first entry and linear in its second. An E_0 -semigroup $\alpha = \{\alpha_t\}_{t \geq 0}$ is a weakly continuous semigroup of unital $*$ -endomorphisms of $B(H)$. Every E_0 -semigroup α is assigned one of three types based on intertwining semigroups called units. A *unit* for α is a strongly continuous semigroup $V = \{V_t\}_{t \geq 0}$ of operators in $B(H)$ such that $\alpha_t(A)V_t = V_tA$ for all $t \geq 0$ and $A \in B(H)$. Let \mathcal{U}_α be the set of units for α . If \mathcal{U}_α is nonempty, we say α is spatial. If, for all $t \geq 0$, the closed linear span of the set $\{U_1(t_1) \cdots U_n(t_n)f : f \in H, t_i \geq 0 \text{ and } U_i \in \mathcal{U}_\alpha \forall i, \sum t_i = t\}$ is H , we say α is completely spatial. If α is completely spatial, we say α is of type I, while if α is spatial but is not completely spatial, we say α is of type II. If α has no units, we say α is of type III. Each spatial E_0 -semigroup is given an index $n \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ which depends on the structure of its units and is invariant under cocycle conjugacy.

We can naturally construct E_0 -semigroups over symmetric and antisymmetric Fock spaces using the right shift semigroup on $K \otimes L^2(0, \infty)$, obtaining the CCR and CAR flows of rank $\dim(K)$. These yield all non-trivial type I E_0 -semigroups in terms of cocycle conjugacy: If α is of type I _{n} (type I, index n) for $n \in \mathbb{N} \cup \{\infty\}$, then α is cocycle conjugate to the CCR flow of rank n (see [2]). The classification of E_0 -semigroups of types II and III is far more complicated, however. Uncountably many examples of both types are known and have been exhibited through greatly differing methods (see, for example, [5], [11], and [12]). Using Bhat's dilation theorem ([3]), Powers showed in [10] that every spatial E_0 -semigroup is induced by the boundary weight map of

Supported by a Graduate Student Fellowship at the University of Pennsylvania and later by the Skirball Foundation via the Center for Advanced Studies in Mathematics at Ben-Gurion University of the Negev.

a CP -flow over $K \otimes L^2(0, \infty)$ for a separable Hilbert space K . He investigated the case when $\dim(K) = 1$ in [11], exhibiting uncountably many mutually non-cocycle conjugate type II_0 E_0 -semigroups using boundary weights over $L^2(0, \infty)$. He also began to explore the case when K is 2-dimensional by combining Schur maps with boundary weights. This approach was generalized to the case when $1 < \dim(K) < \infty$ in [7], where the theory of boundary weight doubles was introduced.

A boundary weight double is a pair (ϕ, ν) , where $\phi : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$ is a q -positive map and ν is a positive boundary weight over $L^2(0, \infty)$ (we write $\nu \in \mathfrak{A}(L^2(0, \infty))_+^+$). If ϕ is unital and ν is normalized and unbounded (in which case we call ν a type II Powers weight), then (ϕ, ν) induces a unital CP -flow over \mathbb{C}^n whose Bhat minimal dilation is a type II_0 E_0 -semigroup. Comparing E_0 -semigroups induced by boundary weight doubles in terms of cocycle conjugacy becomes easier if we focus on the q -pure maps, which are q -positive maps with the smallest possible structure of q -subordinates (see Definition 2.2). The unital q -pure maps which are either rank one or invertible have all been classified in [7]: The unital rank one q -pure maps $\phi : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$ are implemented by faithful states in $M_n(\mathbb{C})^*$, while the unital invertible q -pure maps are a particular class of Schur maps (see Theorems 2.13 and 2.14 for a summary).

Our main goal in this paper is to begin the general classification of all unital q -positive maps $\phi : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$, with the particular aim of finding all such maps which are q -pure. Our second goal is to prove cocycle conjugacy comparison results for boundary weight doubles (ϕ, ν) and (ψ, ν) when ϕ and ψ are not q -pure. We should note that we are only interested in identifying a q -positive map ϕ up to a particular notion of equivalence which we call conjugacy. More precisely, for each q -positive $\phi : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$ and unitary $U \in M_n(\mathbb{C})$, we can form a new q -positive map $\phi_U : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$ by defining $\phi_U(A) = U^* \phi(UAU^*)U$ for all $A \in M_n(\mathbb{C})$. If ϕ is unital and ν is a type II Powers weight of the form $\nu(\sqrt{I - \Lambda(1)}B\sqrt{I - \Lambda(1)}) = (f, Bf)$, then (ϕ, ν) and (ϕ_U, ν) induce cocycle conjugate E_0 -semigroups (Proposition 2.11). In fact, a much stronger result holds: If ϕ is unital and ν is any type II Powers weight, then (ϕ, ν) and (ϕ_U, ν) induce conjugate E_0 -semigroups ([6]). Motivated by this fact, we say that q -positive maps $\phi, \psi : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$ are *conjugate* if $\psi = \phi_U$ for some unitary $U \in M_n(\mathbb{C})$.

Let \mathcal{E}_n be the set of all unital completely positive maps $\Phi : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$ such that $\Phi^2 = \Phi$. This is merely the set of all limits $L_\phi = \lim_{t \rightarrow \infty} t\phi(I + t\phi)^{-1}$ for unital q -positive maps $\phi : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$. This limiting method has already appeared in [7], where it was vital in classifying the unital rank one q -pure maps on $M_n(\mathbb{C})$. We find all elements of \mathcal{E}_2 and \mathcal{E}_3 up to conjugacy. Using this result, we classify the unital q -positive maps $\phi : M_2(\mathbb{C}) \rightarrow M_2(\mathbb{C})$, finding that there is no unital q -positive map $\phi : M_2(\mathbb{C}) \rightarrow M_2(\mathbb{C})$ of rank 3 (Proposition 3.3). Moreover, we find that the only unital q -pure maps $\phi : M_2(\mathbb{C}) \rightarrow M_2(\mathbb{C})$ are either rank one or invertible (Theorem 4.4). We also show that any unital q -positive map $\phi : M_3(\mathbb{C}) \rightarrow M_3(\mathbb{C})$ which annihilates a nonzero positive matrix cannot be q -pure (see Proposition 4.5).

In conclusion, we compare E_0 -semigroups formed by boundary weight doubles (ϕ, ν) and (ψ, ν) in the case that $\phi : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$ ($n \geq 2$) is any unital rank one q -positive map, $\psi : M_k(\mathbb{C}) \rightarrow M_k(\mathbb{C})$ is any unital q -positive map such that L_ψ is a Schur map, and ν is a type II Powers weight of the form $\nu(\sqrt{I - \Lambda(1)}B\sqrt{I - \Lambda(1)}) = (f, Bf)$ (Theorem 5.1). This result substantially generalizes a consequence of Theorems 5.4 and 6.12 of [7].

2. BACKGROUND

2.1. Completely positive and q -positive maps. Let $\phi : B(K) \rightarrow B(H)$ be a linear map. We say that ϕ is *unital* if $\phi(I_K) = I_H$ and *positive* if $\phi(A)$ is positive whenever $A \in B(K)$ is positive. For each $n \in \mathbb{N}$, define $\phi_n : M_n(B(K)) \rightarrow M_n(B(H))$ by

$$\phi_n \begin{pmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{n1} & \cdots & A_{nn} \end{pmatrix} = \begin{pmatrix} \phi(A_{11}) & \cdots & \phi(A_{1n}) \\ \vdots & \ddots & \vdots \\ \phi(A_{n1}) & \cdots & \phi(A_{nn}) \end{pmatrix}.$$

We say that ϕ is completely positive if ϕ_n is positive for all $n \in \mathbb{N}$. If ϕ is completely positive, then $\|\phi\| = \|\phi(I_K)\|$.

We know from a result of Choi (see [4]) that a linear map $\phi : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$ is completely positive if and only if it can be written in the form

$$\phi(A) = \sum_{i=1}^k S_i A S_i^*$$

for some integer $k \leq n^2$ and linearly independent $n \times n$ matrices $\{S_i\}_{i=1}^k$. This result generalizes to normal completely positive maps between $B(K)$ and $B(H)$ for separable Hilbert spaces K and H (see [1]). Denote by $\{e_{ij}\}_{i,j=1}^n$ the set of standard matrix units for $M_n(\mathbb{C})$. Given any $M = \sum_{i,j=1}^n a_{ij} e_{ij} \in M_n(\mathbb{C})$, we can form a linear map $\phi : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$ by defining $\phi(A) = \sum_{i,j} m_{ij} a_{ij} e_{ij}$ for all $A = \sum_{i,j=1}^n a_{ij} e_{ij} \in M_n(\mathbb{C})$. We call this the Schur map corresponding to M , and denote it by the notation $\phi(A) = M \bullet A$. We will frequently use the fact that ϕ is completely positive if and only if M is positive (for a proof, see [9]). By a positive matrix we mean a self-adjoint matrix whose eigenvalues are all nonnegative.

The construction of E_0 -semigroups in [7] (as we will see in Proposition 2.8) required a particular kind of completely positive map:

Definition 2.1. A linear map $\phi : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$ is q -positive if ϕ has no negative eigenvalues and $\phi(I + t\phi)^{-1}$ is completely positive for all $t \geq 0$.

The condition that a completely positive map ϕ must have no negative eigenvalues in order to be q -positive is certainly non-trivial, as completely positive maps with negative eigenvalues exist in abundance. One such example is the Schur map $\phi : M_2(\mathbb{C}) \rightarrow M_2(\mathbb{C})$ defined by

$$\phi \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} a_{11} & -a_{12} \\ -a_{21} & a_{22} \end{pmatrix}.$$

Furthermore, even if ϕ is a completely positive map with no negative eigenvalues, it does not necessarily follow that $\phi(I + t\phi)^{-1}$ is completely positive for all $t \geq 0$. In fact, for each $s \geq 0$, we can construct a completely positive map ϕ which is not q -positive but which still satisfies the condition that $\phi(I + t\phi)^{-1}$ is completely positive for all $0 \leq t \leq s$. For this, let $r \in (1, \sqrt{2}]$ and define a Schur map $\phi_r : M_2(\mathbb{C}) \rightarrow M_2(\mathbb{C})$ by

$$\phi_r \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} a_{11} & \frac{r(1+i)a_{12}}{2} \\ \frac{r(1-i)a_{21}}{2} & a_{22} \end{pmatrix}.$$

In other words, $\phi_r(A) = M \bullet A$ for the positive matrix

$$M = \begin{pmatrix} 1 & \frac{r(1+i)}{2} \\ \frac{r(1-i)}{2} & 1 \end{pmatrix}.$$

We find that $\phi_r(I + t\phi_r)^{-1}(A) = M_t \bullet A$ for all $A \in M_n(\mathbb{C})$ and $t \geq 0$, where

$$M_t = \begin{pmatrix} \frac{1}{1+t} & \frac{r(1+i)}{2+tr(1+i)} \\ \frac{r(1-i)}{2+tr(1-i)} & \frac{1}{1+t} \end{pmatrix}.$$

As noted previously, $A \rightarrow M_t \bullet A$ is completely positive if and only if M_t is a positive matrix. Let λ_1 and λ_2 be the eigenvalues of M_t . Since $\lambda_1 + \lambda_2 = \text{tr}(M_t) > 0$ and $\lambda_1\lambda_2 = \det(M_t)$, M_t is positive if and only if its determinant is nonnegative. A calculation shows that for any given $t \geq 0$, $\det(M_t)$ is nonnegative if and only if $t \leq \frac{2-r^2}{2r(r-1)}$. Therefore, $\phi_r(I + t\phi_r)^{-1}$ ($t \geq 0$) is completely positive if and only if

$$t \leq \frac{2-r^2}{2r(r-1)}.$$

Let $s \geq 0$. The values $(2-r^2)/(2r^2-2r)$ for $r \in (1, \sqrt{2}]$ clearly span $[0, \infty)$, so $s = (2-r_0^2)/(2r_0^2-2r_0)$ for some $r_0 \in (1, \sqrt{2}]$. By the previous paragraph, $\phi_{r_0}(I + t\phi_{r_0})^{-1}$ is completely positive if $0 \leq t \leq s$ but is not completely positive if $t > s$. This example demonstrates that we cannot generally conclude that a map ϕ is q -positive if $\phi(I + t\phi)^{-1}$ is completely positive for all t in some finite interval $J \subset \mathbb{R}_{\geq 0}$, no matter how large J is.

There is a natural order structure for q -positive maps. If $\phi, \psi : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$ are q -positive, we say that ϕ q -dominates ψ (i.e. $\phi \geq_q \psi$) if $\phi(I + t\phi)^{-1} - \psi(I + t\psi)^{-1}$ is completely positive for all $t \geq 0$. As it turns out, for every $s \geq 0$, the map $\phi(I + s\phi)^{-1}$ is q -positive and $\phi \geq_q \phi(I + s\phi)^{-1}$ (Proposition 4.1 of [7]).

Definition 2.2. A q -positive map $\phi : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$ is q -pure if its set of q -subordinates is $\{\phi(I + s\phi)^{-1}\}_{s \geq 0} \cup \{0\}$.

2.2. E_0 -semigroups and CP -flows. A result of Wigner in [13] shows that every one-parameter group $\alpha = \{\alpha_t\}_{t \in \mathbb{R}}$ of $*$ -automorphisms of $B(H)$ is implemented by a strongly continuous unitary group $U = \{U_t\}_{t \in \mathbb{R}}$ in the sense that

$$\alpha_t(A) = U_t A U_t^*$$

for all $A \in B(H)$ and $t \geq 0$. This leads us to ask how to characterize all suitable semigroups of $*$ -endomorphisms of $B(H)$:

Definition 2.3. We say a family $\{\alpha_t\}_{t \geq 0}$ of $*$ -endomorphisms of $B(H)$ is an E_0 -semigroup if:

- (i) $\alpha_{s+t} = \alpha_s \circ \alpha_t$ for all $s, t \geq 0$, and $\alpha_0(A) = A$ for all $A \in B(H)$.
- (ii) For each $f, g \in H$ and $A \in B(H)$, the inner product $(f, \alpha_t(A)g)$ is continuous in t .
- (iii) $\alpha_t(I) = I$ for all $t \geq 0$ (in other words, α is unital).

There are two different conditions under which we think of E_0 -semigroups as equivalent. The first, and stronger condition, is conjugacy, while the second condition, cocycle conjugacy, will be our main focus in comparing E_0 -semigroups.

Definition 2.4. Let α and β be E_0 -semigroups on $B(H_1)$ and $B(H_2)$, respectively. We say that α and β are conjugate if there is a $*$ -isomorphism θ from $B(H_1)$ onto $B(H_2)$ such that $\theta \circ \alpha_t \circ \theta^{-1} = \beta_t$ for all $t \geq 0$.

We say that α and β are cocycle conjugate if α is conjugate to β' , where β' is an E_0 -semigroup on $B(H_2)$ satisfying the following condition: For some strongly continuous family of unitaries $U = \{U_t : t \geq 0\}$ acting on H_2 and satisfying $U_{t+s} = U_t \beta_t(U_s)$ for all $s, t \geq 0$, we have $\beta'_t(A) = U_t \beta_t(A) U_t^*$ for all $A \in B(H_2)$ and $t \geq 0$.

Bhat's dilation theorem from [3] shows that we can obtain E_0 -semigroups from much more general semigroups of completely positive maps called CP -semigroups. A CP -flow is a CP -semigroup acting on $B(K \otimes L^2(0, \infty))$ which is intertwined by the right shift semigroup. More specifically:

Definition 2.5. Let $H = K \otimes L^2(0, \infty)$, which we identify with the space of K -valued measurable functions defined on $(0, \infty)$ which are square integrable. Denote by $U = \{U_t\}_{t \geq 0}$ the right shift semigroup on H , so for all $f \in H$, $x \in (0, \infty)$, and $t \geq 0$, we have $(U_t f)(x) = f(x - t)$ if $x > t$ and $(U_t f)(x) = 0$ otherwise.

A strongly continuous semigroup $\alpha = \{\alpha_t : t \geq 0\}$ of completely positive contractions of $B(H)$ into itself is called a CP -flow if $\alpha_t(A) U_t = U_t A$ for all $t \geq 0$ and $A \in B(H)$.

Unless otherwise specified, we will henceforth write $\{U_t\}_{t \geq 0}$ for the right shift semigroup acting on $K \otimes L^2(0, \infty)$. Special functionals called boundary weights play an important role in constructing CP -flows (see Definition 1.10 of [8] for a more general definition and a detailed discussion):

Definition 2.6. Let $H = K \otimes L^2(0, \infty)$ and define $\Lambda : B(K) \rightarrow B(H)$ by

$$(\Lambda(A)f)(x) = e^{-x} A f(x)$$

for all $A \in B(K)$, $f \in H$, and $x \in (0, \infty)$. We denote by $\mathfrak{A}(H)$ the linear space

$$\mathfrak{A}(H) = \sqrt{I - \Lambda(I_K)} B(H) \sqrt{I - \Lambda(I_K)}$$

and by $\mathfrak{A}(H)_*$ the linear functionals ρ on \mathfrak{A} of the form

$$\rho\left(\sqrt{I - \Lambda(I_K)} A \sqrt{I - \Lambda(I_K)}\right) = \eta(A)$$

for $A \in B(H)$ and $\eta \in B(H)_*$. We call such functionals boundary weights.

We can associate to every CP -flow α a boundary weight map $\rho \rightarrow \omega(\rho)$ from $B(K)_*$ to $\mathfrak{A}(H)_*$ which is related to α in the following manner. Let R_α be the resolvent

$$R_\alpha(A) = \int_0^\infty e^{-t} \alpha_t(A) dt$$

of α , and define $\Gamma : B(H) \rightarrow B(H)$ by $\Gamma(A) = \int_0^\infty e^{-t} U_t A U_t^* dt$ for all $A \in B(H)$. Using hats to denote the predual mappings, we have

$$\hat{R}_\alpha(\tau) = \hat{\Gamma}\left(\omega(\hat{\Lambda}\tau) + \tau\right)$$

for all $\tau \in B(H)_*$. If we let $\rho \rightarrow \omega_t(\rho)$ be the truncated boundary weight maps

$$(1) \quad \omega_t(\rho)(A) = \omega(\rho)\left(U_t U_t^* A U_t U_t^*\right),$$

for all $t > 0$ and $A \in B(H)$, then $\omega_t(I + \hat{\Lambda}\omega_t)^{-1}$ is a completely positive contraction from $B(K)_*$ into $B(H)_*$ for every $t > 0$.

Having seen that every CP -flow has an associated boundary weight map, we naturally ask when a given map $\rho \rightarrow \omega(\rho)$ from $B(K)_*$ to $\mathfrak{A}(H)_*$ is the boundary weight map of a CP -flow. The answer is that if $\rho \rightarrow \omega(\rho)$ is a completely positive map from $B(K)_*$ into $\mathfrak{A}(H)_*$ satisfying $\omega(\rho)(I - \Lambda(I_K)) \leq \rho(I_K)$ for all positive $\rho \in B(K)_*$, and if $\omega_t(I + \hat{\Lambda}\omega_t)^{-1}$ is a completely positive contraction of $B(K)_*$ into $B(H)_*$ for every $t > 0$, then $\rho \rightarrow \omega(\rho)$ is the boundary weight map of a unique CP -flow over K (see Theorem 3.3 of [11]). This CP -flow is unital if and only if $\omega(\rho)(I - \Lambda(I_K)) = \rho(I_K)$ for all $\rho \in B(K)_*$.

Suppose α is a CP -flow over \mathbb{C} . We identify its boundary weight map with the single positive boundary weight $\omega := \omega(1) \in \mathfrak{A}(L^2(0, \infty))_*$. From above, ω has the form

$$\omega(\sqrt{I - \Lambda(1)}B\sqrt{I - \Lambda(1)}) = \sum_{i=1}^n (f_i, Bf_i)$$

for some mutually orthogonal nonzero L^2 -functions $\{f_k\}_{k=1}^n$ and unique $n \in \mathbb{N} \cup \{\infty\}$. If α is unital, then $\sum_{i=1}^n \|f_i\|^2 = 1$, and we say ω is *normalized*. We say ω is *bounded* if there exists an $r > 0$ such that $|\omega(A)| \leq r\|A\|$ for all $A \in \mathfrak{A}(L^2(0, \infty))$. Otherwise, we say ω is *unbounded*. From [10], we know that if ω is bounded, then the Bhat dilation α^d of α is of type I_n , while if ω is unbounded, then α^d is of type II_0 . Being type II_0 means that α_t^d is a proper $*$ -endomorphism for all $t > 0$ and that α^d has exactly one unit $V = \{V_t\}_{t \geq 0}$ up to exponential scaling. In other words, a semigroup of bounded operators $W = \{W_t\}_{t \geq 0}$ acting on H is a unit for α^d if and only if, for some $\lambda \in \mathbb{C}$, we have $W_t = e^{\lambda t}V_t$ for all $t \geq 0$. This paragraph leads us to make the definition:

Definition 2.7. A normalized positive boundary weight $\nu \in \mathfrak{A}(L^2(0, \infty))_*$ is said to be a type I (respectively, type II) Powers weight if ν is bounded (respectively, unbounded).

If $\dim(K) > 1$, we can naturally construct type II_0 E_0 -semigroups by combining type II Powers weights with q -positive maps acting on $M_n(\mathbb{C})$ (Proposition 3.2 and Corollary 3.3 of [7]):

Proposition 2.8. Let $H = \mathbb{C}^n \otimes L^2(0, \infty)$. Let $\phi : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$ be a unital completely positive map with no negative eigenvalues, and let ν be a type II Powers weight. Let $\Omega_\nu : \mathfrak{A}(H) \rightarrow M_n(\mathbb{C})$ be the map that sends $A = (A_{ij}) \in M_n(\mathfrak{A}(L^2(0, \infty))) \cong \mathfrak{A}(H)$ to the matrix $(\nu(A_{ij})) \in M_n(\mathbb{C})$.

Then the map $\rho \rightarrow \omega(\rho)$ from $M_n(\mathbb{C})^*$ into $\mathfrak{A}(H)_*$ defined by

$$\omega(\rho)(A) = \rho\left(\phi(\Omega_\nu(A))\right)$$

is the boundary weight map of a unital CP -flow α over \mathbb{C}^n if and only if ϕ is q -positive, in which case the Bhat minimal dilation α^d of α is a type II_0 E_0 -semigroup.

In the notation of this proposition, we say α^d is the E_0 -semigroup induced by the boundary weight double (ϕ, ν) . There is no ambiguity in doing so, since α^d is unique up to conjugacy by Bhat's theorem. Suppose that (ϕ, ν) and (ψ, μ) are boundary weight doubles which induce E_0 -semigroups α^d and β^d . When are α^d and β^d cocycle conjugate? We have a partial answer, and it involves the following definition:

Definition 2.9. Let $\phi : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$ and $\psi : M_k(\mathbb{C}) \rightarrow M_k(\mathbb{C})$ be q -positive maps. We say a linear map $\gamma : M_{n \times k}(\mathbb{C}) \rightarrow M_{n \times k}(\mathbb{C})$ a corner from ϕ to ψ if the map

$$\Upsilon \begin{pmatrix} A_{n \times n} & B_{n \times k} \\ C_{k \times n} & D_{k \times k} \end{pmatrix} = \begin{pmatrix} \phi(A_{n \times n}) & \gamma(B_{n \times k}) \\ \gamma^*(C_{k \times n}) & \psi(D_{k \times k}) \end{pmatrix}$$

is completely positive. We say γ is a q -corner if Υ is q -positive. A q -corner γ is called hyper maximal if, whenever

$$\Upsilon \geq_q \Upsilon' = \begin{pmatrix} \phi' & \gamma \\ \gamma^* & \psi' \end{pmatrix} \geq_q 0,$$

we have $\Upsilon = \Upsilon'$.

The main result of [7] with regard to comparing E_0 -semigroups induced by boundary weight doubles (ϕ, ν) and (ψ, ν) is the following, which unfortunately requires ν to have a very specific form:

Proposition 2.10. Let $\phi : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$ and $\psi : M_k(\mathbb{C}) \rightarrow M_k(\mathbb{C})$ be unital q -positive maps, and let ν be a type II Powers weight of the form

$$\nu(\sqrt{I - \Lambda(1)}B\sqrt{I - \Lambda(1)}) = (f, Bf).$$

The boundary weight doubles (ϕ, ν) and (ψ, ν) induce cocycle conjugate E_0 -semigroups if and only if there is a hyper maximal q -corner from ϕ to ψ .

Let $\phi : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$ be unital and q -positive, and let $U \in M_n(\mathbb{C})$ be unitary. Define $\phi_U : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$ by

$$\phi_U(A) = U^* \phi(UAU^*)U$$

for all $A \in M_n(\mathbb{C})$. It is straightforward to show that ϕ_U is also unital and q -positive. We note that the map $\gamma : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$ defined by $\gamma(A) = \phi(UA^*)U$ is a hyper maximal q -corner from ϕ to ϕ_U . Indeed, it is easy to check that γ is a q -corner from ϕ to ϕ_U (see Proposition 4.5 of [7]). To see that γ is hyper maximal, we observe that if

$$\begin{pmatrix} \phi & \gamma \\ \gamma^* & \phi_U \end{pmatrix} \geq_q \begin{pmatrix} \phi' & \gamma \\ \gamma^* & \phi'_U \end{pmatrix} \geq_q 0,$$

then $\phi'(I) \leq I$ and $\phi'_U(I) \leq I$, yet

$$\begin{pmatrix} \phi'(I) & \gamma(U) \\ \gamma^*(U^*) & \phi'_U(I) \end{pmatrix} = \begin{pmatrix} \phi'(I) & U \\ U^* & \phi'_U(I) \end{pmatrix} \geq 0,$$

hence $\phi'(I) = \phi'_U(I) = I$. But $\phi - \phi'$ and $\phi_U - \phi'_U$ are completely positive, so

$$\|\phi - \phi'\| = \|\phi(I) - \phi'(I)\| = 0 = \|\phi_U(I) - \phi'_U(I)\| = \|\phi_U - \phi'_U\|,$$

thus $\phi = \phi'$ and $\phi_U = \phi'_U$. This shows that γ is hyper maximal, whereby Proposition 2.10 gives us the following:

Proposition 2.11. Let $\phi : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$ be unital and q -positive, and let $U \in M_n(\mathbb{C})$ be unitary. If ν is a type II Powers weight of the form

$$\nu(\sqrt{I - \Lambda(1)}B\sqrt{I - \Lambda(1)}) = (f, Bf),$$

then (ϕ, ν) and (ϕ_U, ν) induce cocycle conjugate E_0 -semigroups.

In fact, if ν is an arbitrary type II Powers weight, then (ϕ, ν) and (ϕ_U, ν) induce conjugate E_0 -semigroups ([6]). We will not use this result here, except as justification for the following definition.

Definition 2.12. Let $\phi : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$ be q -positive. We say ψ is conjugate to ϕ if $\psi = \phi_U$ for some unitary $U \in M_n(\mathbb{C})$.

In other words, ψ is conjugate to ϕ if and only if there is a $*$ -isomorphism $\theta : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$ such that $\theta \circ \psi \circ \theta^{-1} = \phi$. This is analogous to the notion of conjugacy for E_0 -semigroups and is appropriate in light of the preceding paragraph. We recall the classification of all q -pure maps $\phi : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$ which are rank one or invertible, along with the main cocycle conjugacy results of [7] (see Lemma 5.2 and Theorem 5.4 of [7]):

Theorem 2.13. A unital rank one linear map $\phi : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$ is q -positive if and only if it has the form $\phi(A) = \rho(A)I$ for some state $\rho \in M_n(\mathbb{C})^*$. Such a map ϕ is q -pure if and only if ρ is faithful.

Let ϕ and ψ be unital rank one q -pure maps on $M_n(\mathbb{C})$ and $M_k(\mathbb{C})$, respectively, and let ν be a type II Powers weight of the form $\nu(\sqrt{I - \Lambda(1)}B\sqrt{I - \Lambda(1)}) = (f, Bf)$. The boundary weight doubles (ϕ, ν) and (ψ, ν) induce cocycle conjugate E_0 -semigroups if and only if $n = k$ and ϕ is conjugate to ψ .

Furthermore, if ν and μ are type II Powers weights and ϕ and ψ are rank one unital q -pure maps on $M_n(\mathbb{C})$ and $M_k(\mathbb{C})$, respectively, then (ϕ, ν) and (ψ, μ) cannot induce cocycle conjugate E_0 -semigroups unless there is a corner γ from ϕ to ψ such that $\|\gamma\| = 1$ (Lemma 5.3 of [7]). A consequence of this result is that if $n > 1$, then none of the E_0 -semigroups induced by boundary weight doubles (ϕ, ν) for unital rank one q -pure $\phi : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$ and ν of the form $\nu(\sqrt{I - \Lambda(1)}B\sqrt{I - \Lambda(1)}) = (f, Bf)$ are cocycle conjugate to any of the E_0 -semigroups constructed by Powers in the case that $\dim(K) = 1$ in [11]. However, for q -pure maps that are invertible rather than rank one, the opposite holds (Theorems 6.11 and 6.12 of [7]):

Theorem 2.14. An invertible unital linear map $\phi : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$ is q -positive if and only if ϕ^{-1} is conditionally negative, and ϕ is q -pure if and only if ϕ^{-1} is of the form

$$\phi^{-1}(A) = A + YA + AY^*$$

for some $Y \in M_n(\mathbb{C})$ with $Y = -Y^*$ and $\text{tr}(Y) = 0$. Equivalently, ϕ is q -pure if and only if it is conjugate to a Schur map ψ that satisfies

$$\psi(a_{jk}e_{jk}) = \begin{pmatrix} \frac{a_{jk}}{1+i(\lambda_j-\lambda_k)}e_{jk} & \text{if } j < k \\ a_{jk}e_{jk} & \text{if } j = k \\ \frac{a_{jk}}{1-i(\lambda_j-\lambda_k)}e_{jk} & \text{if } j > k \end{pmatrix}$$

for all $j, k = 1, \dots, n$ and all $A = \sum a_{ij}e_{ij} \in M_n(\mathbb{C})$, where $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ and $\sum_{j=1}^n \lambda_j = 0$.

If ν is a type II Powers weight of the form $\nu(\sqrt{I - \Lambda(1)}B\sqrt{I - \Lambda(1)}) = (f, Bf)$, then the E_0 -semigroup induced by (ϕ, ν) is cocycle conjugate to the E_0 -semigroup induced by $(\iota_{\mathbb{C}}, \nu)$ for $\iota_{\mathbb{C}}$ the identity map on \mathbb{C} (this is the E_0 -semigroup induced by ν in the sense of [11]).

3. \mathcal{E}_n AND THE LIMITING MAP L_ϕ

Suppose $\phi : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$ is a q -positive map and $\|t\phi(I + t\phi)^{-1}\| < 1$ for all $t > 0$. In [7], we saw that we could form a limit $L_\phi = \lim_{t \rightarrow \infty} t\phi(I + t\phi)^{-1}$. This limiting process was the key to classifying the rank one q -pure maps acting on $M_n(\mathbb{C})$. We begin this section by revisiting L_ϕ :

Lemma 3.1. *Suppose $\phi : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$ is a non-zero q -positive map such that $\|t\phi(I + t\phi)^{-1}\| < 1$ for all $t \geq 0$. Then the maps $t\phi(I + t\phi)^{-1}$ have a unique limit L_ϕ as $t \rightarrow \infty$, and $\|L_\phi\| = 1$. Furthermore, L_ϕ is completely positive, $L_\phi \circ \phi = \phi \circ L_\phi = \phi$, $\text{range}(L_\phi) = \text{range}(\phi)$, $\text{nullspace}(\phi) = \text{nullspace}(L_\phi)$, and $L_\phi^2 = L_\phi$.*

Proof: A compactness argument shows that since $\|t\phi(I + t\phi)^{-1}\| < 1$ for all $t > 0$, the maps $t\phi(I + t\phi)^{-1}$ have some norm limit L_ϕ as $t \rightarrow \infty$, where $\|L_\phi\| \leq 1$. To see this limit is unique, we let $M \in M_{2n}(\mathbb{C})$ be the matrix for ϕ with respect to some orthonormal basis of $M_n(\mathbb{C})$ and note that the entries of $tM(I + tM)^{-1}$ are (necessarily bounded) rational functions of t and thus each have unique limits. L_ϕ is completely positive since it is the norm limit of completely positive maps.

For every $t > 0$, let $M_t = (I + t\phi)/t$, so $M_t \rightarrow \phi$ as $t \rightarrow \infty$. Given any $A \in M_n(\mathbb{C})$, we find

$$\begin{aligned} \phi(L_\phi(A)) &= \lim_{t \rightarrow \infty} M_t(t\phi(I + t\phi)^{-1})(A) = \lim_{t \rightarrow \infty} \left(\frac{I + t\phi}{t} \right) t\phi(I + t\phi)^{-1}(A) \\ &= \phi(I + t\phi)(I + t\phi)^{-1}(A) = \phi(A), \end{aligned}$$

so $\phi \circ L_\phi = \phi$. But L_ϕ commutes with ϕ by construction, so $L_\phi \circ \phi = \phi$, hence $\text{range}(\phi) \subseteq \text{range}(L_\phi)$. Trivially $\text{range}(L_\phi) \subseteq \text{range}(\phi)$, so $\text{range}(\phi) = \text{range}(L_\phi)$, whereby the fact that $L_\phi \circ \phi = \phi$ implies that L_ϕ fixes its range, hence $L_\phi^2 = L_\phi$. Since L_ϕ is a nonzero contraction, we have $\|L_\phi\| = 1$. To finish the proof, we need only show that ϕ and L_ϕ have the same nullspace. The fact that $\text{nullspace}(L_\phi) \subseteq \text{nullspace}(\phi)$ follows trivially from the established equality $\phi = \phi \circ L_\phi$. On the other hand, if $\phi(A) = 0$, then $(I + t\phi)^{-1}(A) = A$ for all $t \geq 0$, hence

$$L_\phi(A) = \lim_{t \rightarrow \infty} t\phi(I + t\phi)^{-1}(A) = \lim_{t \rightarrow \infty} t\phi(A) = 0.$$

□

Any unital q -positive $\phi : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$ satisfies the conditions of the above lemma, since for all $t \geq 0$ we have

$$\|t\phi(I + t\phi)^{-1}\| = \|t\phi(I + t\phi)^{-1}(I)\| = \frac{t}{1+t}.$$

Definition 3.2. *For each $n \in \mathbb{N}$, let \mathcal{E}_n be the set of all unital completely positive maps $\Phi : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$ such that $\Phi^2 = \Phi$.*

If $\phi : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$ is unital and q -positive, then $L_\phi \in \mathcal{E}_n$ by Lemma 3.1. On the other hand, let $\Phi \in \mathcal{E}_n$ be arbitrary. Since $\Phi^2 = \Phi$, it follows that $I + t\Phi$ is invertible for all $t \geq 0$ and $t\Phi(I + t\Phi)^{-1} = (t/(1+t))\Phi$, so Φ is q -positive and $\Phi = L_\Phi$. Therefore,

$$\mathcal{E}_n = \{L_\phi \mid \phi : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C}), \phi(I) = I, \text{ and } \phi \geq_q 0\}.$$

Note that membership in \mathcal{E}_n is invariant under conjugacy: If $\Phi \in \mathcal{E}_n$ and $U \in M_n(\mathbb{C})$ is unitary, then Φ_U is unital and completely positive by construction, and $\Phi_U^2 = \Phi_U$

since

$$\begin{aligned}\Phi_U^2(A) &= \Phi_U(U^*\Phi(UAU^*)U) = U^*\Phi\left[U\left(U^*\Phi(UAU^*)U\right)U^*\right]U \\ &= U^*\Phi^2(UAU^*)U = U^*\Phi(UAU^*)U = \Phi_U(A)\end{aligned}$$

for all $A \in M_n(\mathbb{C})$.

The rest of this section is devoted to classifying the elements of \mathcal{E}_2 and \mathcal{E}_3 up to conjugacy. As we will see in the next section, this is a key step in classifying all unital q -positive maps $\phi : M_2(\mathbb{C}) \rightarrow M_2(\mathbb{C})$ and in showing that a large class of q -positive maps acting on $M_3(\mathbb{C})$ cannot be q -pure.

Remark: It is possible for a unital completely positive map $\phi : M_2(\mathbb{C}) \rightarrow M_2(\mathbb{C})$ to have rank 3. For example, define $\phi : M_2(\mathbb{C}) \rightarrow M_2(\mathbb{C})$ by

$$\phi(A) = \frac{1}{3} \begin{pmatrix} 2a_{11} + a_{22} & a_{12} + a_{21} \\ a_{12} + a_{21} & a_{11} + 2a_{22} \end{pmatrix}.$$

We see that ϕ has rank 3, since

$$\text{range}(\phi) = \left\{ \begin{pmatrix} a & b \\ b & c \end{pmatrix} : a, b, c \in \mathbb{C} \right\}.$$

Furthermore, ϕ is completely positive since it is the sum of completely positive maps, as

$$\phi(A) = \frac{1}{3} \left(A + SAS^* + D(A) \right),$$

where $S = e_{12} + e_{21}$ and D is the diagonal map $D(A) = a_{11}e_{11} + a_{22}e_{22}$. However, it turns out that ϕ is not q -positive. In fact, we will see from our classification of \mathcal{E}_2 that no unital q -positive map ϕ acting on $M_2(\mathbb{C})$ can have rank 3:

Proposition 3.3. *Let $\Phi : M_2(\mathbb{C}) \rightarrow M_2(\mathbb{C})$ be a unital linear map. Then $\Phi \in \mathcal{E}_2$ if and only if, up to conjugacy, Φ has one of the forms below:*

- (i) $\Phi(A) = \rho(A)I$ for all $A \in M_2(\mathbb{C})$, where $\rho \in M_2(\mathbb{C})^*$ is a state.
- (ii) $\Phi(A) = a_{11}e_{11} + a_{22}e_{22}$ for all $A = \sum_{i,j=1}^2 a_{ij}e_{ij} \in M_2(\mathbb{C})$.
- (iii) $\Phi(A) = A$ for all $A \in M_2(\mathbb{C})$.

Consequently, if $\phi : M_2(\mathbb{C}) \rightarrow M_2(\mathbb{C})$ is unital and q -positive, then $\text{rank}(\phi) \neq 3$.

Proof: By inspection, maps (i) through (iii) (and therefore their conjugates) are in \mathcal{E}_2 . On the other hand, suppose Φ is an element of \mathcal{E}_2 . If Φ has rank one, then it trivially has the form (i). If $\text{rank}(\Phi) \geq 2$, then $\Phi(I) = I$ and $\Phi(A_1) = A_1$ for some A_1 linearly independent from I . Since Φ is completely positive and thus self-adjoint in the sense that $\Phi(A^*) = \Phi(A)^*$ for all A , we have $\Phi(A_1 + A_1^*) = A_1 + A_1^*$ and $\Phi(i(A_1 - A_1^*)) = i(A_1 - A_1^*)$. A quick exercise in linear algebra shows that the self-adjoint matrices $A_1 + A_1^*$ and $i(A_1 - A_1^*)$ cannot both be multiples of I , whereby we conclude that $\Phi(M) = M$ for some self-adjoint $M \in M_n(\mathbb{C})$ linearly independent from I .

Letting U be a unitary matrix such that $U^*MU = D$ for some diagonal matrix D , we note that D is linearly independent from I . We observe that $\Phi_U(I) = I$ and $\Phi_U(D) = U^*\Phi(UDU^*)U = U^*MU = D$, which implies $\Phi_U(e_{11}) = e_{11}$ and $\Phi_U(e_{22}) = e_{22}$. We claim that $\Phi_U(e_{12}) = be_{12}$ for some $b \in \mathbb{C}$. Indeed, write

$$\Phi_U(e_{12}) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Since Φ_U is 2-positive, we have

$$0 \leq \begin{pmatrix} \Phi_U(e_{11}) & \Phi_U(e_{12}) \\ \Phi_U(e_{21}) & \Phi_U(e_{22}) \end{pmatrix} = \begin{pmatrix} 1 & 0 & a & b \\ 0 & 0 & c & d \\ \bar{a} & \bar{c} & 0 & 0 \\ \bar{b} & \bar{d} & 0 & 1 \end{pmatrix}.$$

Positivity of the above matrix implies $a = c = d = 0$, hence $\Phi_U(e_{12}) = be_{12}$ and $\Phi_U(e_{21}) = \bar{b}e_{21}$. Therefore Φ_U is merely the Schur mapping

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \rightarrow \begin{pmatrix} a_{11} & ba_{12} \\ \bar{b}a_{21} & a_{22} \end{pmatrix}.$$

Since $(\Phi_U)^2 = \Phi_U$ we have $b^2 = b$, so either $b = 0$ (in which case Φ_U has the form (ii)) or $b = 1$ (in which case Φ_U is the identity map (iii)).

For the final statement of the theorem, we note that if $\phi : M_2(\mathbb{C}) \rightarrow M_2(\mathbb{C})$ is unital and q -positive, then $L_\phi \in \mathcal{E}_2$ and $\text{rank}(\phi) = \text{rank}(L_\phi)$, so $\text{rank}(L_\phi) \in \{1, 2, 4\}$ by what we have just shown.

□

We turn our attention to classifying the elements of \mathcal{E}_3 up to conjugacy. Our task is made much easier by the fact that each of its elements with rank greater than one must destroy or fix a rank one projection:

Lemma 3.4. *Suppose $\Phi \in \mathcal{E}_3$ and $\text{rank}(\Phi) > 1$. If Φ does not annihilate any nonzero projections, then Φ fixes some rank one projection E .*

Proof: Since $\text{rank}(\Phi) \geq 2$ and Φ fixes its range, Φ fixes some $M \in M_3(\mathbb{C})$ linearly independent from I . Arguing as in the proof of Proposition 3.3, we may assume $M = M^*$, and of course we may assume $\|M\| = 1$. Since M is self-adjoint and has norm one, we know that at least one of the numbers 1 and -1 is an eigenvalue of M . Therefore, replacing M with $-M$ if necessary, we may assume that 1 is an eigenvalue of M . Diagonalizing M by a unitary $U \in M_3(\mathbb{C})$ so that the eigenvalues of $D := UMU^*$ are listed in decreasing order, we have

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix}$$

where $1 \geq \lambda_1 \geq \lambda_2$. Note that $\lambda_2 \neq 1$ since $D \neq I$. Since Φ_U fixes (respectively, annihilates) a projection P if and only if Φ fixes (respectively, annihilates) the projection UPU^* , it suffices to show that Φ_U fixes a rank one projection.

Note that $\Phi_U(I) = I$ and $\Phi_U(D) = D$, so

$$(2) \quad \Phi_U(I - D) = I - D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 - \lambda_1 & 0 \\ 0 & 0 & 1 - \lambda_2 \end{pmatrix} \geq 0.$$

If $\lambda_1 = 1$, then Φ_U fixes e_{33} and the lemma follows. If $\lambda_1 \neq 1$, then we let $b = (1 - \lambda_2)/(1 - \lambda_1) > 0$. By complete positivity of Φ_U and equation (2),

$$(3) \quad 0 \leq \Phi_U(e_{22}) \leq \Phi_U(e_{22} + be_{33}) = e_{22} + be_{33}.$$

We also note that

$$\Phi_U(D - \lambda_2 I) = D - \lambda_2 I = \begin{pmatrix} 1 - \lambda_2 & 0 & 0 \\ 0 & \lambda_1 - \lambda_2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \geq 0.$$

If $\lambda_1 = \lambda_2$, then Φ_U fixes e_{11} and the lemma follows. If $\lambda_1 \neq \lambda_2$, we let $c = (1 - \lambda_2)/(\lambda_1 - \lambda_2) > 0$ and note that

$$(4) \quad 0 \leq \Phi_U(e_{22}) \leq \Phi_U(ce_{11} + e_{22}) = ce_{11} + e_{22}.$$

Equation (3) implies that the 11 entry of $\Phi_U(e_{22})$ is zero, while equation (4) implies that the 33 entry of $\Phi_U(e_{22})$ is zero. Therefore, $\Phi_U(e_{22}) = \lambda e_{22}$ for some $\lambda \geq 0$. Since $\Phi_U^2 = \Phi_U$ we have $\lambda \in \{0, 1\}$, whereby the fact that Φ_U does not annihilate any nonzero projections implies $\lambda = 1$. Thus, Φ_U fixes e_{22} , so Φ fixes the rank one projection $Ue_{22}U^*$. □

Before proceeding further, we will need the following two standard results regarding completely positive maps:

Lemma 3.5. *Let K be a separable Hilbert space, and let $\phi : B(K) \rightarrow B(K)$ be a normal completely positive map. If $\phi(E) = 0$ for some projection E , then $\phi(A) = \phi(FAF)$ for all $A \in B(K)$, where $F = I - E$.*

From [1], we know that ϕ can be written in the form $\phi(A) = \sum_{i=1}^p S_i A S_i^*$ for some $p \in \mathbb{N} \cup \{\infty\}$ and operators $\{S_i\}_{i=1}^p$ in $B(K)$. We note that

$$0 = \phi(E) = \sum_{i=1}^p S_i E S_i^* = \sum_{i=1}^p S_i E E S_i^* = \sum_{i=1}^p (S_i E)(S_i E)^*,$$

so $S_i E = 0 = E S_i^*$ for all i . Therefore, $\phi(EAE) = \phi(EAF) = \phi(FAE) = 0$ for all $A \in B(K)$, hence

$$\begin{aligned} \phi(A) &= \phi((E + F)A(E + F)) \\ &= \phi(EAE) + \phi(EAF) + \phi(FAE) + \phi(FAF) \\ &= \phi(FAF). \end{aligned}$$
□

Lemma 3.6. *Let K be a separable Hilbert space, and let $\phi : B(K) \rightarrow B(K)$ be a normal and unital completely positive map. Suppose ϕ fixes a projection E . Then*

$$\phi(A) = E\phi(EAE)E + E\phi(EAF)F + F\phi(FAE)E + F\phi(FAF)F$$

for all A , where $F = I - E$.

Proof: By hypothesis, we can write ϕ in the form $\phi(A) = \sum_{i=1}^p S_i A S_i^*$. Since $\phi(I) = I$ and $\phi(E) = E$, we have $\phi(F) = \phi(I - E) = I - E = F$. Therefore, $S_i E S_i^* \leq E$ and $S_i F S_i^* \leq F$ for all i . Note that $E S_i F = F S_i^* E = 0$ for all i , since

$$(E S_i F)(E S_i F)^* = E S_i F F S_i^* E = E(S_i F S_i^*)E \leq E F E = 0.$$

An analogous argument shows that $FS_iE = ES_i^*F = 0$ for all i . Writing

$$\phi(A) = (E + F)\phi\left((E + F)A(E + F)\right)(E + F)$$

and expanding the right hand side using the above makes most of the terms vanish, yielding the result. \square

With the previous three lemmas in hand, we are able to classify the elements of \mathcal{E}_3 in two steps.

Lemma 3.7. *Let $\Phi : M_3(\mathbb{C}) \rightarrow M_3(\mathbb{C})$ be a unital map such that $\Phi(E) = 0$ for some nonzero projection E . Then $\Phi \in \mathcal{E}_3$ if and only if, up to conjugacy, Φ has one of the following forms for some $\lambda \in [0, 1]$:*

$$(I) \quad \Phi(A) = \left(\lambda a_{22} + (1 - \lambda)a_{33}\right)I;$$

$$(II) \quad \Phi(A) = \begin{pmatrix} \lambda a_{22} + (1 - \lambda)a_{33} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix};$$

$$(III) \quad \Phi(A) = \begin{pmatrix} \lambda a_{22} + (1 - \lambda)a_{33} & 0 & 0 \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{pmatrix};$$

Proof: The backward direction follows from inspection of the maps (I) through (III). For the forward direction, suppose $\Phi \in \mathcal{E}_3$ and $\Phi(E) = 0$. Let E' be any rank one subprojection of E , observing that $\Phi(E') = 0$. Unitarily diagonalizing E' so that $Z^*E'Z = e_{11}$, we have $\Phi_Z(e_{11}) = 0$. By Lemma 3.5 it follows that $\Phi_Z(A) = \Phi_Z(FAF)$ for all $A \in M_3(\mathbb{C})$, where $F = I - e_{11}$. Replacing Φ_Z with Φ (as we are only concerned with Φ up to conjugacy), we write

$$\Phi(A) = \begin{pmatrix} \tau_1(A) & \tau_2(A) & \tau_3(A) \\ \tau_2^*(A) & & \\ \tau_3^*(A) & [\Psi(A)] & \end{pmatrix}$$

for some linear functionals τ_j ($j = 1, 2, 3$) and some map $\Psi : M_3(\mathbb{C}) \rightarrow M_2(\mathbb{C})$. But $\Phi(A) = \Phi(FAF)$ for all $A \in M_3(\mathbb{C})$, so $\Phi(e_{1j}) = \Phi(e_{j1}) = 0$ for $j = 1, 2, 3$. Therefore, for every $A \in M_3(\mathbb{C})$, $\Psi(A)$ and each $\tau_j(A)$ depend only on the bottom right 2×2 minor of A . In other words, if we let

$$G = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in M_{2 \times 3}(\mathbb{C})$$

and define $\rho_j \in M_2(\mathbb{C})^*$ ($j = 1, 2, 3$) and $\psi : M_2(\mathbb{C}) \rightarrow M_2(\mathbb{C})$ by $\rho_j(B) = \tau(G^*BG)$ and $\psi(B) = \Psi(G^*BG)$, then for all $A \in M_3(\mathbb{C})$,

$$(5) \quad \Phi(A) = \begin{pmatrix} \rho_1(GAG^*) & \rho_2(GAG^*) & \rho_3(GAG^*) \\ \rho_2^*(GAG^*) & & \\ \rho_3^*(GAG^*) & [\psi(GAG^*)] & \end{pmatrix}.$$

Note that $\psi(B) = G\Phi(G^*BG)G^*$ for all $B \in M_2(\mathbb{C})$, so ψ is completely positive, and ψ is unital since for the identity matrix $I_2 \in M_2(\mathbb{C})$, we have

$$\psi(I_2) = G\Phi(G^*G)G^* = G\Phi(F)G^* = GIG^* = I_2.$$

Furthermore, $\psi^2 = \psi$, since

$$\begin{aligned} \psi^2(B) &= \psi\left(G\Phi(G^*BG)G^*\right) = G\Phi\left(G^*G\Phi(G^*BG)G^*G\right)G^* \\ &= G\Phi\left(F\Phi(G^*BG)F\right)G^* = G\Phi\left(\Phi(G^*BG)\right)G^* \\ &= G\Phi(G^*BG)G^* = \psi(B), \end{aligned}$$

where for the fourth equality we used the fact that $\Phi(A) = \Phi(FAF)$ for all $A \in M_3(\mathbb{C})$. Therefore, $\psi \in \mathcal{E}_2$, whereby Proposition 3.3 implies that $\text{rank}(\psi) \neq 3$.

Case (i): If $\text{rank}(\psi) = 1$, then ψ is of the form $\psi(B) = \rho(B)I_2$, where $\rho \in M_2(\mathbb{C})^*$ satisfies $\rho(I_2) = 1$. By equation (5) and the fact that $\Phi^2 = \Phi$, we have

$$(6) \quad \rho_j(GAG^*) = \rho_j(\psi(GAG^*)) = \rho_j(\rho(GAG^*)I_2) = \rho(GAG^*)\rho_j(I_2)$$

for every $A \in M_3(\mathbb{C})$ and $j = 1, 2, 3$. But $\Phi(I) = I$, so $\rho_1(I_2) = 1$ while $\rho_2(I_2) = \rho_3(I_2) = 0$, so equation (6) implies $\rho_1 = \rho$ and $\rho_2 = \rho_3 \equiv 0$.

Since $\rho \in M_2(\mathbb{C})^*$ is a state, there is some $\lambda \in [0, 1]$ and a unitary matrix $S \in M_2(\mathbb{C})$ such that $\rho(SBS^*) = \lambda b_{11} + (1 - \lambda)b_{22}$ for all $B \in M_2(\mathbb{C})$. Therefore,

$$\rho\left(S\begin{bmatrix} GAG^* \\ S^* \end{bmatrix}S^*\right) = \lambda a_{22} + (1 - \lambda)a_{33}$$

for all $A \in M_3(\mathbb{C})$. Letting

$$R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & [S] \\ 0 & \end{pmatrix},$$

we see that Φ_R has the form (I).

Case (ii): If $\text{rank}(\psi) = 2$, then Lemma 3.3 implies that for some 2×2 unitary V , ψ_V is the diagonal map

$$\psi_V \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} b_{11} & 0 \\ 0 & b_{22} \end{pmatrix}.$$

Let $U \in M_3(\mathbb{C})$ be the 3×3 unitary matrix

$$U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & [V] \\ 0 & \end{pmatrix}.$$

Then $\Phi_U(e_{11}) = U^*\Phi(Ue_{11}U^*)U = U^*\Phi(e_{11})U = 0$ and $G\Phi_U(G^*BG)G^* = \psi_V(B)$ for all $B \in M_2(\mathbb{C})$. Therefore, Φ_U has the form below for some linear functionals ρ'_j , $j = 1, 2, 3$:

$$(7) \quad \Phi_U(A) = \begin{pmatrix} \rho'_1(GAG^*) & \rho'_2(GAG^*) & \rho'_3(GAG^*) \\ \rho'_2(GAG^*) & a_{22} & 0 \\ \rho'_3(GAG^*) & 0 & a_{33} \end{pmatrix}.$$

Replacing Φ_U with Φ and erasing the primes on the functionals for simplicity of notation, we have

$$\Phi(A) = \begin{pmatrix} \rho_1(GAG^*) & \rho_2(GAG^*) & \rho_3(GAG^*) \\ \rho_2^*(GAG^*) & a_{22} & 0 \\ \rho_3^*(GAG^*) & 0 & a_{33} \end{pmatrix}.$$

Positivity of the matrices $\Phi(e_{22})$ and $\Phi(e_{33})$ yields

$$(8) \quad \rho_3 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = 0 \quad \text{and} \quad \rho_2 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = 0,$$

respectively. Since Φ is unital we have

$$(9) \quad \rho_3 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \rho_2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 0,$$

and combining equations (8) and (9) gives us

$$(10) \quad \rho_2(D) = \rho_3(D) = 0 \quad \text{for all diagonal matrices } D \in M_2(\mathbb{C}).$$

For $j = 1, 2, 3$, the fact that $\Phi^2(e_{23}) = \Phi(e_{23})$ implies

$$(11) \quad \rho_j \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \rho_j \left(\psi \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right) = \rho_j \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0,$$

and similarly, since $\Phi^2(e_{32}) = \Phi(e_{32})$, we have

$$(12) \quad \rho_j \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = 0.$$

From equations (10), (11), and (12), we have $\rho_2 = \rho_3 \equiv 0$ and

$$(13) \quad \rho_1 \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \rho_1 \begin{pmatrix} b_{11} & 0 \\ 0 & b_{22} \end{pmatrix}$$

for all $B \in M_2(\mathbb{C})$. From equation (13) and the fact that Φ is unital, there is some $\lambda \in [0, 1]$ such that

$$\rho_1(GAG^*) = \lambda a_{22} + (1 - \lambda)a_{33}$$

for all $A \in M_3(\mathbb{C})$, hence Φ satisfies (II).

Case (iii): If $\text{rank}(\psi) = 4$, then ψ is the identity map by Lemma 3.3, so

$$\Phi(A) = \begin{pmatrix} \rho_1(GAG^*) & \rho_2(GAG^*) & \rho_3(GAG^*) \\ \rho_2^*(GAG^*) & a_{22} & a_{23} \\ \rho_3^*(GAG^*) & a_{32} & a_{33} \end{pmatrix}.$$

Arguing as we did in the case that $\text{rank}(\psi) = 2$, we see that $\rho_2(D) = \rho_3(D) = 0$ for all diagonal matrices $D \in M_2(\mathbb{C})$, so for $j = 2, 3$,

$$(14) \quad \rho_j \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \rho_j \begin{pmatrix} 0 & b_{12} \\ b_{21} & 0 \end{pmatrix}$$

for all $B \in M_2(\mathbb{C})$. For each c on the unit circle S^1 , let

$$w_c = \rho_2 \begin{pmatrix} 0 & c \\ \bar{c} & 0 \end{pmatrix}, \quad z_c = \rho_3 \begin{pmatrix} 0 & c \\ \bar{c} & 0 \end{pmatrix}.$$

Applying Φ to the family of positive 3×3 matrices $\{M_c\}_{c \in S^1}$ defined by $M_c = e_{22} + ce_{23} + \bar{c}e_{32} + e_{33}$, we find

$$\begin{aligned} 0 &\leq \det(\Phi(M_c)) = -|w_c|^2 - |z_c|^2 + 2\operatorname{Re}(cw_c\bar{z}_c) \\ &= -|cw_c|^2 - |z_c|^2 + 2\operatorname{Re}(cw_c\bar{z}_c) = -|cw_c - z_c|^2, \end{aligned}$$

hence $cw_c = z_c$ for all $c \in S^1$. This gives us

$$(15) \quad c^2\rho_2 \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + |c|^2\rho_2 \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = c\rho_3 \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \bar{c}\rho_3 \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

for all $c \in S^1$. Applying (15) to $c = 1$ and $c = -1$ yields

$$(16) \quad \rho_j \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = -\rho_j \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

for $j = 2, 3$. Letting

$$b = \rho_2 \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad d = \rho_3 \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

we rewrite (15) as

$$c^2b - |c|^2b = cd - \bar{c}d.$$

Applying this to $c = i$ and $c = -i$ yields $b = d = 0$, whereby

$$\rho_j \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = 0 \quad \text{and thus} \quad \rho_j \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = 0 \quad \text{by (16) for } j = 2, 3.$$

We conclude from (14) that $\rho_2 = \rho_3 \equiv 0$, hence Φ has the form

$$\Phi(A) = \begin{pmatrix} \rho_1(GAG^*) & 0 & 0 \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{pmatrix}.$$

Since ρ_1 is a state on $M_2(\mathbb{C})$, we know that for some unitary $Y \in M_2(\mathbb{C})$ and $\lambda \in [0, 1]$, we have

$$\rho_1\left(Y \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} Y^*\right) = \lambda b_{11} + (1 - \lambda)b_{22}$$

for all $B \in M_2(\mathbb{C})$, so for every $A \in M_3(\mathbb{C})$,

$$\rho_1\left(Y[GAG^*]Y^*\right) = \lambda a_{22} + (1 - \lambda)a_{33}.$$

Letting

$$X = \begin{pmatrix} 1 & 0 & 0 \\ 0 & & \\ 0 & [Y] & \end{pmatrix},$$

we observe that Φ_X has the form (III).

□

Lemma 3.8. *Suppose $\Phi : M_3(\mathbb{C}) \rightarrow M_3(\mathbb{C})$ is a linear map which does not annihilate any projections and satisfies $\operatorname{rank}(\Phi) > 1$. Then $\Phi \in \mathcal{E}_3$ if and only if, up to conjugacy, it has one of the following forms for all $A \in M_3(\mathbb{C})$:*

$$(IV) \quad \Phi(A) = \begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix};$$

$$(V) \quad \Phi(A) = \begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{pmatrix};$$

$$(VI) \quad \Phi(A) = \begin{pmatrix} a_{11} & 0 & 0 \\ 0 & \lambda a_{22} + (1-\lambda)a_{33} & 0 \\ 0 & 0 & \lambda a_{22} + (1-\lambda)a_{33} \end{pmatrix}, \quad \lambda \in (0, 1);$$

$$(VII) \quad \Phi(A) = A.$$

Proof: The backward direction follows from inspection of the maps (IV) through (VII). Assume the hypotheses of the forward direction. By Lemma 3.4, Φ fixes a rank one projection E . Note that $U^*EU = e_{11}$ for some unitary $U \in M_3(\mathbb{C})$, so

$$\Phi_U(e_{11}) = e_{11}.$$

Therefore, we may assume that $E = e_{11}$ and $\Phi(e_{11}) = e_{11}$. Let $F = I - E = e_{22} + e_{33}$. For some functionals $\tau_2, \tau_3 \in M_3(\mathbb{C})^*$ and some linear map $\Psi : M_3(\mathbb{C}) \rightarrow M_2(\mathbb{C})$, we have

$$\Phi(A) = \begin{pmatrix} a_{11} & \tau_2(A) & \tau_3(A) \\ \tau_2^*(A) & & \\ \tau_3^*(A) & [\Psi(A)] & \end{pmatrix}.$$

However, by Lemma 3.6, Φ satisfies

$$\Phi(A) = E\Phi(EAE)E + E\Phi(EAF)F + F\Phi(FAE)E + F\Phi(FAF)F,$$

so

$$(17) \quad \Psi(A) = \Psi(FAF), \quad \tau_j(A) = \tau_j(EAF) = \tau_j(a_{12}e_{12} + a_{13}e_{13})$$

for all $A \in M_3(\mathbb{C})$ and $j = 2, 3$. Let

$$G = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in M_{2 \times 3}(\mathbb{C}) \quad \text{and} \quad J = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \in M_{3 \times 1}(\mathbb{C}).$$

Defining $\psi : M_2(\mathbb{C}) \rightarrow M_2(\mathbb{C})$ and $\rho_j \in M_{1 \times 2}(\mathbb{C})^*$ ($j = 2, 3$) by $\psi(B) = \Psi(G^*BG)$ and $\rho_j(C) = \tau_j(JCG)$ for all $B \in M_2(\mathbb{C})$ and $C \in M_{1 \times 2}(\mathbb{C})$, we see that Φ has the form

$$(18) \quad \Phi(A) = \begin{pmatrix} a_{11} & \rho_2 \begin{pmatrix} a_{12} & a_{13} \end{pmatrix} & \rho_3 \begin{pmatrix} a_{12} & a_{13} \end{pmatrix} \\ \rho_2^* \begin{pmatrix} a_{21} \\ a_{31} \end{pmatrix} & & \\ \rho_3^* \begin{pmatrix} a_{21} \\ a_{31} \end{pmatrix} & \left[\psi \begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix} \right] & \end{pmatrix}$$

for all $A \in M_3(\mathbb{C})$.

From equation (18) and the fact that $\Phi^2 = \Phi$, we have $\psi^2 = \psi$ and $\psi(I_2) = I_2$ for the 2×2 identity matrix I_2 . Moreover, ψ is completely positive since $\psi(B) =$

$G\Phi(G^*BG)G^*$ for all $B \in M_2(\mathbb{C})$. Therefore, $\psi \in \mathcal{E}_2$, so $\text{rank}(\psi) \in \{1, 2, 4\}$ by Proposition 3.3.

Case (i): If ψ has rank one, then it has the form

$$\psi \begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix} = \rho \begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix} I_2,$$

where ρ is faithful since Φ does not annihilate any nonzero projections. For all $A \in M_3(\mathbb{C})$ we have

$$\Phi(A) = \begin{pmatrix} a_{11} & \rho_2 \begin{pmatrix} a_{12} & a_{13} \end{pmatrix} & \rho_3 \begin{pmatrix} a_{12} & a_{13} \end{pmatrix} \\ \rho_2^* \begin{pmatrix} a_{21} \\ a_{31} \end{pmatrix} & \left[\rho \begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix} I_2 \right] \\ \rho_3^* \begin{pmatrix} a_{21} \\ a_{31} \end{pmatrix} & \end{pmatrix}.$$

Let C be the matrix

$$C = \begin{pmatrix} \rho_2(1 & 0) & \rho_2(0 & 1) \\ \rho_3(1 & 0) & \rho_3(0 & 1) \end{pmatrix}.$$

Since $\Phi^2(e_{12}) = \Phi(e_{12})$ and $\Phi^2(e_{13}) = \Phi(e_{13})$, we have $C^2 = C$.

If $C = 0$, then we repeat a familiar argument: Since ρ is faithful and

$$\rho \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1,$$

we know that for some 2×2 unitary T and $\lambda \in (0, 1)$,

$$\rho \left(T \begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix} T^* \right) = \lambda a_{22} + (1 - \lambda) a_{33}$$

for all $A \in FM_3(\mathbb{C})F$. Letting

$$Z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & [T] \\ 0 & \end{pmatrix},$$

we see that

$$\Phi_Z(A) = \begin{pmatrix} a_{11} & 0 & 0 \\ 0 & \lambda a_{22} + (1 - \lambda) a_{33} & 0 \\ 0 & 0 & \lambda a_{22} + (1 - \lambda) a_{33} \end{pmatrix}$$

for all $A \in M_3(\mathbb{C})$, so Φ_Z has the form (VI).

Now suppose $\text{rank}(C) \geq 1$. Since $C^2 = C$, C fixes a unit vector \vec{x} ,

$$\vec{x} = \begin{pmatrix} a \\ b \end{pmatrix}, \quad |a|^2 + |b|^2 = 1.$$

In other words,

$$\begin{pmatrix} a \\ b \end{pmatrix} = C \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \rho_2(a & b) \\ \rho_3(a & b) \end{pmatrix}.$$

Letting

$$A = \begin{pmatrix} 1 \\ \bar{a} \\ \bar{b} \end{pmatrix} \begin{pmatrix} 1 & a & b \end{pmatrix} = \begin{pmatrix} 1 & a & b \\ \bar{a} & |a|^2 & \bar{a}b \\ \bar{b} & a\bar{b} & |b|^2 \end{pmatrix} = \begin{pmatrix} 1 & a & b \\ \bar{a} & [P] \\ \bar{b} & \end{pmatrix} \geq 0,$$

we have

$$\Phi(A) = \begin{pmatrix} 1 & a & b \\ \bar{a} & \rho(P) & 0 \\ \bar{b} & 0 & \rho(P) \end{pmatrix} \geq 0,$$

hence $0 \leq \det(A) = \rho(P)(\rho(P) - |a|^2 - |b|^2) = \rho(P)(\rho(P) - 1)$. But $0 < \rho(P) \leq 1$ since P is a rank one projection, so $\rho(P) = 1$. Therefore, Φ annihilates the rank one projection

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & [I_2 - P] \\ 0 & \end{pmatrix},$$

contradicting our assumption that Φ does not destroy any nonzero projections.

Case (ii): If $\text{rank}(\psi) = 2$, then by Proposition 3.3, ψ_V is the diagonal map for some unitary $V \in M_2(\mathbb{C})$. Letting

$$S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & [V] \\ 0 & \end{pmatrix},$$

we see $\Phi_S(e_{11}) = S^*\Phi(Se_{11}S^*)S = S^*\Phi(e_{11})S^* = e_{11}$ and $\psi_V(B) = G\Phi_S(G^*BG)G^*$ for all $B \in M_2(\mathbb{C})$. Since Φ_S fixes e_{11} and does not annihilate any nonzero projections, we may argue as we did earlier in the proof (using Lemma 3.6) to conclude that for some functionals ρ'_2 and ρ'_3 acting on $M_{1 \times 2}(\mathbb{C})$, Φ_S has the form

$$\Phi_S(A) = \begin{pmatrix} a_{11} & \rho'_2 \begin{pmatrix} a_{12} & a_{13} \end{pmatrix} & \rho'_3 \begin{pmatrix} a_{12} & a_{13} \end{pmatrix} \\ \rho'^*_2 \begin{pmatrix} a_{21} \\ a_{31} \end{pmatrix} & a_{22} & 0 \\ \rho'^*_3 \begin{pmatrix} a_{21} \\ a_{31} \end{pmatrix} & 0 & a_{33} \end{pmatrix}$$

Replacing Φ_S with Φ and erasing the primes from the functionals ρ'_2 and ρ'_3 , we continue our argument.

Now

$$\Phi \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & \rho_2(1 \ 0) & \rho_3(1 \ 0) \\ \frac{\rho_2(1 \ 0)}{\rho_3(1 \ 0)} & 1 & 0 \\ \frac{\rho_2(1 \ 0)}{\rho_3(1 \ 0)} & 0 & 0 \end{pmatrix} \geq 0,$$

hence $\rho_3(1 \ 0) = 0$. Similarly, positivity of $\Phi(e_{11} + e_{13} + e_{31} + e_{33})$ implies $\rho_2(0 \ 1) = 0$. It follows that for some $z_2, z_3 \in \mathbb{C}$, we have $\rho_2(a_{12} \ a_{13}) = z_2 a_{12}$ and $\rho_3(a_{12} \ a_{13}) = z_3 a_{13}$ for all $(a_{12} \ a_{13}) \in M_{1 \times 2}(\mathbb{C})$. Since $\Phi^2 = \Phi$ we have $z_j^2 = z_j$, so $z_j \in \{0, 1\}$ for $j = 2, 3$. Therefore, Φ is the Schur map $\Phi(A) = M \bullet A$, where

$$M = \begin{pmatrix} 1 & z_2 & z_3 \\ z_2 & 1 & 0 \\ z_3 & 0 & 1 \end{pmatrix} \geq 0.$$

If $z_2 = z_3 = 0$, then Φ has the form (IV). If $z_2 = 1$, then by positivity of M we have $z_3 = 0$, and we note that for the unitary matrix $U = e_{13} + e_{22} + e_{31}$, Φ_U has the form of (V). On the other hand, if $z_3 = 1$ then $z_2 = 0$ by positivity of M . Letting $V = e_{12} + e_{21} + e_{33}$, we observe that Φ_V has the form of (V).

Case (iii): If ψ is the identity map, we may repeat the same argument we just used to show that for some $z_2, z_3 \in \{0, 1\}$, we have $\rho_2(a_{12} \ a_{13}) = z_2 a_{12}$ and $\rho_3(a_{12} \ a_{13}) = z_3 a_{13}$ for all $(a_{12} \ a_{13}) \in M_{1 \times 2}(\mathbb{C})$. Therefore, $\Phi(A) = N \bullet A$ for all $A \in M_3(\mathbb{C})$, where

$$N = \begin{pmatrix} 1 & z_2 & z_3 \\ z_2 & 1 & 1 \\ z_3 & 1 & 1 \end{pmatrix} \geq 0.$$

From positivity of N , we conclude that either $z_2 = z_3 = 1$ (i.e. Φ is the identity map (VII)) or $z_2 = z_3 = 0$ (in which case Φ has the form (V)).

□

Lemmas 3.7 and 3.8 give us the following:

Theorem 3.9. *A linear map $\Phi : M_3(\mathbb{C}) \rightarrow M_3(\mathbb{C})$ is in \mathcal{E}_3 if and only if Φ has the form $\Phi(A) = \rho(A)I$ for some faithful state $\rho \in M_3(\mathbb{C})^*$ or, up to conjugacy, Φ is one of the maps (I) through (VII).*

Proof: The only case not covered by Lemmas 3.7 and 3.8 is when Φ is a rank one map which does not annihilate any nonzero projections. It is clear that such a map Φ is in \mathcal{E}_3 if and only if it is of the form $\Phi(A) = \rho(A)I$ for a faithful state ρ .

□

Corollary 3.10. *Let $\phi : M_3(\mathbb{C}) \rightarrow M_3(\mathbb{C})$ be unital and q -positive. Then ϕ has rank 1, 2, 3, 4, 5, or 9.*

4. CLASSIFICATION OF UNITAL q -POSITIVE AND q -PURE MAPS ON $M_2(\mathbb{C})$

If unital q -positive maps ϕ and ψ are conjugate, then their limits L_ϕ and L_ψ are naturally conjugate as well:

Lemma 4.1. *Suppose $\phi : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$ is q -positive and $\|\phi(I + t\phi)^{-1}\| < 1$ for all $t > 0$, and let $U \in M_n(\mathbb{C})$ be unitary. Then $L_{(\phi_U)} = (L_\phi)_U$.*

Proof: We know from Proposition 4.5 of [7] that

$$t\phi_U(I + t\phi_U)^{-1}(A) = U^*\phi(I + t\phi)^{-1}(UAU^*)U$$

for all $t > 0$ and $A \in M_n(\mathbb{C})$, so

$$\begin{aligned} L_{\phi_U}(A) &= \lim_{t \rightarrow \infty} t\phi_U(I + t\phi_U)^{-1}(A) = \lim_{t \rightarrow \infty} tU^*\phi(I + t\phi)^{-1}(UAU^*)U \\ &= U^* \left[\lim_{t \rightarrow \infty} t\phi(I + t\phi)^{-1}(UAU^*) \right] U = U^* L_\phi(UAU^*)U = (L_\phi)_U(A). \end{aligned}$$

□

Proposition 4.2. *Let $\phi : M_2(\mathbb{C}) \rightarrow M_2(\mathbb{C})$ be a unital linear map of rank 2. Then ϕ is q -positive if and only if, for some unitary $U \in M_2(\mathbb{C})$ and numbers $\lambda \in (0, 1]$, $\lambda' \in [0, 1)$ with $\lambda > \lambda'$, we have*

$$(19) \quad \phi_U(A) = \begin{pmatrix} \lambda a_{11} + (1 - \lambda)a_{22} & 0 \\ 0 & \lambda' a_{11} + (1 - \lambda')a_{22} \end{pmatrix}$$

for all $A = \sum_{i,j=1}^2 a_{ij}e_{ij} \in M_2(\mathbb{C})$.

Proof: For the forward direction, assume that ϕ is q -positive. It follows from Proposition 3.3 that for some unitary $U \in M_2(\mathbb{C})$, $(L_\phi)_U$ is the diagonal map

$$(L_\phi)_U(A) = a_{11}e_{11} + a_{22}e_{22}.$$

But $(L_\phi)_U = L_{(\phi_U)}$ by Lemma 4.1 and $\text{range}(\phi_U) = \text{range}(L_{\phi_U})$ by Lemma 3.1, hence $\text{range}(\phi_U) = \text{span}\{e_{11}, e_{22}\}$. Therefore, for some positive functionals $\rho_1, \rho_2 \in M_2(\mathbb{C})^*$,

$$\phi_U(A) = \rho_1(A)e_{11} + \rho_2(A)e_{22}$$

for all $A \in M_2(\mathbb{C})$, where ρ_1 and ρ_2 are states since $\phi_U(I) = I$. Since $\text{nullspace}(\phi_U) = \text{nullspace}(L_{\phi_U})$ by Lemma 3.1, we have $\phi_U(e_{12}) = \phi_U(e_{21}) = 0$, so $\rho_j(e_{12}) = \rho_j(e_{21}) = 0$ for $j = 1, 2$. Therefore, there are numbers $\lambda, \lambda' \in [0, 1]$ such that

$$(20) \quad \rho_1(A) = \lambda a_{11} + (1 - \lambda)a_{22}, \quad \rho_2(A) = \lambda' a_{11} + (1 - \lambda')a_{22}$$

for all $A \in M_2(\mathbb{C})$. Let $Q = \lambda - \lambda'$, and for every $t \geq 0$, let

$$D_t = 1 + t(1 + Q) + t^2Q.$$

To prove the forward direction, it suffices to show that $Q > 0$, since it will then automatically follow that $\lambda \in (0, 1]$ and $\lambda' \in [0, 1)$. For $j = 1, 2$, let $\nu_j \in M_2(\mathbb{C})^*$ be the functional $\nu_j(A) = a_{jj}$. If $t \geq 0$ and $D_t \neq 0$, then a straightforward computation shows that $I + t\phi_U$ is invertible and

$$(I + t\phi_U)^{-1}(A) = A - \mu_{1,t}(A)e_{11} - \mu_{2,t}(A)e_{22}$$

for all $A \in M_2(\mathbb{C})$, where μ_1 and μ_2 are the functionals

$$(21) \quad \mu_{1,t} = \frac{t(\lambda + tQ)\nu_1 + t(1 - \lambda)\nu_2}{D_t}, \quad \mu_{2,t} = \frac{t\lambda'\nu_1 + t(1 - \lambda' + tQ)\nu_2}{D_t}.$$

It follows that

$$(22) \quad t\phi_U(I + t\phi_U)^{-1}(A) = \mu_{1,t}(A)e_{11} + \mu_{2,t}(A)e_{22}$$

for all $A \in M_2(\mathbb{C})$. If $Q = 0$, then $\text{rank}(\phi) = 1$ by (20), contradicting our assumption that $\text{rank}(\phi) = 2$. If $Q < 0$, then $D_{t_0} = 0$ for some $t_0 > 0$. Since $\|t\phi_U(I + t\phi_U)^{-1}\| < 1$ for all $t > 0$, the numerators of $\mu_{1,t}$ and $\mu_{2,t}$ must both approach zero as $t \rightarrow t_0$. With regard to $\mu_{1,t}$, this means that either $\lambda = 1$ (contradicting our assumption that $Q < 0$) or

$$\nu_2 = -\frac{\lambda + t_0Q}{1 - \lambda}\nu_1,$$

which is clearly impossible. Thus $Q > 0$, proving the forward direction.

Now assume the hypotheses of the backward direction. For every $t > 0$, we have $D_t > 0$, so $I + t\phi_U$ is invertible and $t\phi_U(I + t\phi_U)^{-1}$ has the form (22), where $\mu_{1,t}$ and $\mu_{2,t}$ are positive linear functionals by (21). Therefore, ϕ_U (and thus ϕ) is q -positive. \square

Theorem 4.3. *Let $\phi : M_2(\mathbb{C}) \rightarrow M_2(\mathbb{C})$ be a unital linear map. Then ϕ is q -positive if and only if it satisfies one of the following:*

- (i) $\phi(A) = \rho(A)I$ for all $A \in M_2(\mathbb{C})$, where $\rho \in M_2(\mathbb{C})^*$ is a state.
- (ii) For some $\lambda \in (0, 1]$ and $\lambda' \in [0, 1)$ with $\lambda > \lambda'$, ϕ is conjugate to the map ψ defined by

$$\psi(A) = \begin{pmatrix} \lambda a_{11} + (1 - \lambda)a_{22} & 0 \\ 0 & \lambda' a_{11} + (1 - \lambda')a_{22} \end{pmatrix}.$$

(iii) $\phi = \psi^{-1}$, where $\psi : M_2(\mathbb{C}) \rightarrow M_2(\mathbb{C})$ is a unital conditionally negative map.

Proof: By Proposition 3.3, we may assume that ϕ has rank 1, 2, or 4. From Proposition 4.2 and Theorems 2.13 and 2.14, conditions (i), (ii), and (iii) are the necessary and sufficient conditions for q -positivity of unital linear maps $\phi : M_2(\mathbb{C}) \rightarrow M_2(\mathbb{C})$ of rank 1, 2, and 4, respectively. \square

Now that we have every unital q -positive $\phi : M_2(\mathbb{C}) \rightarrow M_2(\mathbb{C})$, we find that the only such maps which are q -pure are rank one or invertible.

Theorem 4.4. *A unital linear map $\phi : M_2(\mathbb{C}) \rightarrow M_2(\mathbb{C})$ is q -pure if and only if it satisfies one of the following:*

- (i) $\phi(A) = \rho(A)I$ for all $A \in M_2(\mathbb{C})$, where $\rho \in M_2(\mathbb{C})^*$ is a faithful state;
- (ii) For some $\lambda \in \mathbb{R}$, ϕ is conjugate to the Schur map ψ defined by

$$\psi(A) = \begin{pmatrix} a_{11} & \frac{a_{12}}{1+i\lambda} \\ \frac{a_{21}}{1-i\lambda} & a_{22} \end{pmatrix}$$

for all $A \in M_2(\mathbb{C})$.

Proof: By Theorems 2.13 and 2.14, conditions (i) and (ii) are the necessary and sufficient conditions for a unital linear map of rank 1 or 4 to be q -pure. Suppose that ϕ is a unital q -positive map of rank 2, so by Theorem 4.3, it is conjugate to a map of the form (19). Since q -purity is invariant under conjugacy (Proposition 4.5 of [7]), it suffices to assume ϕ has the form (19) and show that ϕ is not q -pure. Defining ν_1 and ν_2 as in the proof of Proposition 4.2, we recall that for every $t \geq 0$, we have $t\phi(I + t\phi)^{-1}(A) = \mu_{1,t}(A)e_{11} + \mu_{2,t}(A)e_{22}$, where $Q := \lambda - \lambda' > 0$ and

$$\mu_{1,t} = t \frac{(\lambda + tQ)\nu_1 + (1 - \lambda)\nu_2}{1 + t(1 + Q) + t^2Q}, \quad \mu_{2,t} = t \frac{\lambda'\nu_1 + (1 - \lambda' + tQ)\nu_2}{1 + t(1 + Q) + t^2Q}.$$

Define $\Phi : M_2(\mathbb{C}) \rightarrow M_2(\mathbb{C})$ by

$$\Phi(A) = \frac{Q\nu_1(A)}{1 - \lambda'}e_{11}.$$

For every $t \geq 0$ and $A \in M_2(\mathbb{C})$, we have

$$(I + t\Phi)^{-1}(A) = A - \frac{tQ\nu_1(A)}{1 - \lambda' + tQ}e_{11}$$

and

$$\Phi(I + t\Phi)^{-1}(A) = \frac{Q\nu_1(A)}{1 - \lambda' + tQ}e_{11},$$

thus $\Phi \geq_q 0$. Straightforward computations show that $\phi - \Phi$ is completely positive and that for all $t > 0$, we have

$$(23) \quad \left(\phi(I + t\phi)^{-1} - \Phi(I + t\Phi)^{-1} \right)(A) = \eta_{1,t}(A)e_{11} + \frac{\mu_{2,t}(A)}{t}e_{22}$$

for all $A \in M_2(\mathbb{C})$, where

$$\eta_{1,t}(A) = \frac{(\lambda + tQ)\nu_1(A) + (1 - \lambda)\nu_2(A)}{1 + t(1 + Q) + t^2Q} - \frac{Q\nu_1(A)}{1 - \lambda' + tQ}.$$

Note that for every $t > 0$,

$$\begin{aligned} \eta_{1,t} \geq 0 &\iff \frac{(\lambda + tQ)\nu_1 + (1 - \lambda)\nu_2}{1 + t(1 + Q) + t^2Q} \geq \frac{Q\nu_1}{1 - \lambda' + tQ} \\ &\iff \frac{(\lambda + t(\lambda - \lambda'))\nu_1 + (1 - \lambda)\nu_2}{1 + t(1 + \lambda - \lambda') + t^2(\lambda - \lambda')} \geq \frac{(\lambda - \lambda')\nu_1}{1 - \lambda' + t(\lambda - \lambda')} \\ &\iff (1 - \lambda)\lambda'\nu_1 + \left((1 - \lambda)(1 - \lambda' + t(\lambda - \lambda'))\right)\nu_2 \geq 0. \end{aligned}$$

The coefficients of ν_1 and ν_2 are nonnegative in the above expression, so $\eta_{1,t}$ is a positive linear functional for all $t > 0$, hence $\phi \geq_q \Phi$ by (23). But $\text{rank}(\Phi) = 1$ while $\text{rank}(\phi(I + s\phi)^{-1}) = 2$ for all $s \geq 0$, so ϕ is not q -pure.

□

Proposition 4.5. *If $\phi : M_3(\mathbb{C}) \rightarrow M_3(\mathbb{C})$ is a unital q -positive map and $\phi(R) = 0$ for some $R \succcurlyeq 0$, then ϕ is not q -pure.*

Proof: If $\phi(R) = 0$ for some nonzero positive $R \in M_3(\mathbb{C})$, then ϕ annihilates a rank one projection E . Letting $U \in M_3(\mathbb{C})$ be any unitary matrix such that $U^*EU = e_{11}$, we have $\phi_U(e_{11}) = 0$. Since q -purity is invariant under conjugacy, we may replace ϕ_U with ϕ and continue our argument. Since $\phi(e_{11}) = 0$ we have $L_\phi(e_{11}) = 0$. Replacing L_ϕ (and therefore ϕ) with one of its conjugates if necessary, we conclude L_ϕ has one of the forms (I) through (III). Since ϕ and L_ϕ have the same range and nullspace, it follows that

$$(24) \quad \text{range}(\phi) \subseteq \text{span}\{e_{11}, e_{22}, e_{23}, e_{32}, e_{33}\} \quad \text{and} \quad \phi(e_{1j}) = \phi(e_{j1}) = 0 \text{ for } j = 1, 2, 3.$$

Let $F = e_{22} + e_{33}$. Line (24) and Lemma 3.5 imply that for some state τ and some map $\Psi : M_3(\mathbb{C}) \rightarrow M_2(\mathbb{C})$, ϕ has the form

$$\phi(A) = \begin{pmatrix} \tau(A) & 0 & 0 \\ 0 & [\Psi(A)] \\ 0 & \end{pmatrix},$$

where $\tau(A) = \tau(FAF)$ and $\Psi(A) = \Psi(FAF)$ for all $A \in M_3(\mathbb{C})$. Letting

$$G = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and defining $\rho \in M_2(\mathbb{C})^*$ and $\psi : M_2(\mathbb{C}) \rightarrow M_2(\mathbb{C})$ by $\rho(B) = \tau(G^*BG)$ and $\psi(B) = \Psi(G^*BG)$ for all $B \in M_2(\mathbb{C})$, we observe that ϕ has the form

$$\phi(A) = \begin{pmatrix} \rho(GAG^*) & 0 & 0 \\ 0 & [\psi(GAG^*)] \\ 0 & \end{pmatrix}.$$

Note that ψ has no negative eigenvalues. Indeed, suppose that $\psi(B) = \lambda B$ for some $\lambda < 0$ and $B \in M_2(\mathbb{C})$. Let $c = \rho(B)$. We see

$$\phi\left(\frac{c}{\lambda} e_{11} + B\right) = 0 + \phi(B) = ce_{11} + \lambda B = \lambda\left(\frac{c}{\lambda} e_{11} + B\right),$$

contradicting the fact that ϕ has no negative eigenvalues.

Define $\phi' : M_3(\mathbb{C}) \rightarrow M_3(\mathbb{C})$ by $\phi'(A) = F\phi(A)F$ for all $A \in M_3(\mathbb{C})$, so

$$\phi'(A) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & [\psi(GAG^*)] \\ 0 & \end{pmatrix}.$$

We claim that ϕ' is q -positive. Note that since ψ has no negative eigenvalues, the same is true of ϕ' . Since ϕ' commutes with $(I + t\phi')^{-1}$ for all $t \geq 0$, we have

$$(25) \quad \begin{aligned} \phi'(I + t\phi')^{-1}(A) &= (I + t\phi')^{-1}\phi'(A) = (I + t\phi')^{-1}\phi'(FAF) \\ &= \phi'(I + t\phi')^{-1}(FAF) \end{aligned}$$

for all $A \in M_3(\mathbb{C})$, and similarly, $\phi(I + t\phi)^{-1}(A) = \phi(I + t\phi)^{-1}(FAF)$.

Let $A \in M_3(\mathbb{C})$. For some 2×2 matrix B , we have

$$(26) \quad (I + t\phi')^{-1}(FAF) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & [B] \\ 0 & \end{pmatrix}, \quad \phi'(I + t\phi')^{-1}(FAF) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & [\psi(B)] \\ 0 & \end{pmatrix}$$

while for some $d \in \mathbb{C}$, we have

$$(27) \quad (I + t\phi)^{-1}(FAF) = \begin{pmatrix} d & 0 & 0 \\ 0 & [B] \\ 0 & \end{pmatrix}, \quad \phi(I + t\phi)^{-1}(FAF) = \begin{pmatrix} \rho(B) & 0 & 0 \\ 0 & [\psi(B)] \\ 0 & \end{pmatrix}.$$

Combining (25), (26), and (27), we find that for all $A \in M_3(\mathbb{C})$ and $t \geq 0$:

$$(28) \quad \begin{aligned} \phi'(I + t\phi')^{-1}(A) &= \phi'(I + t\phi')^{-1}(FAF) = F\left(\phi(I + t\phi)^{-1}(FAF)\right)F \\ &= F\left(\phi(I + t\phi)^{-1}(A)\right)F. \end{aligned}$$

This shows that ϕ' is q -positive. Furthermore, from equation (28) and the fact that

$$\phi(I + t\phi)^{-1}(A) = e_{11}\left(\phi(I + t\phi)^{-1}(A)\right)e_{11} + F\left(\phi(I + t\phi)^{-1}(A)\right)F$$

for all $A \in M_3(\mathbb{C})$, we find that

$$\phi(I + t\phi)^{-1}(A) - \phi'(I + t\phi')^{-1}(A) = e_{11}\left(\phi(I + t\phi)^{-1}(FAF)\right)e_{11}.$$

Since the last line is the composition of completely positive maps for every $t \geq 0$, we have $\phi \geq_q \phi'$. Finally, we note that $e_{11}\phi'(I)e_{11} = 0$, whereas for every $s \geq 0$,

$$e_{11}\left(\phi(I + s\phi)^{-1}(I)\right)e_{11} = e_{11}\left(\frac{1}{1+s}I\right)e_{11} = \frac{1}{1+s}e_{11}.$$

Therefore, ϕ' is not equal to $\phi(I + s\phi)^{-1}$ for any $s \geq 0$, so ϕ is not q -pure. □

5. A COCYCLE CONJUGACY RESULT

Let ν be a type II Powers weight of the form $\nu(\sqrt{I - \Lambda(1)}B\sqrt{I - \Lambda(1)}) = (f, Bf)$. Suppose $\phi : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$ ($n \geq 2$) and $\psi : M_k(\mathbb{C}) \rightarrow M_k(\mathbb{C})$ are unital and q -positive, where $\text{rank}(\phi) = 1$ and ψ is invertible. We have seen that if ϕ and ψ are q -pure, then they are fundamentally “different” in the sense that (ϕ, ν) and (ψ, ν) induce non-cocycle conjugate E_0 -semigroups (a consequence of Theorems 2.13 and 2.14). We now find that the previous sentence holds if we remove the assumption

that ϕ and ψ are q -pure. In fact, we may replace the assumption that ψ is invertible with the much weaker assumption that L_ψ is a Schur map:

Theorem 5.1. *Let $\phi : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$ ($n \geq 2$) and $\psi : M_k(\mathbb{C}) \rightarrow M_k(\mathbb{C})$ be unital q -positive maps. Suppose that ϕ has rank one and that L_ψ is a Schur map. Let ν be a type II Powers weight of the form $\nu(\sqrt{I - \Lambda(1)}B\sqrt{I - \Lambda(1)}) = (f, Bf)$.*

Then (ϕ, ν) and (ψ, ν) induce non-cocycle conjugate E_0 -semigroups.

Proof: Let α^d and β^d be the E_0 -semigroups induced by (ϕ, ν) and (ψ, ν) , respectively. Suppose there is a nonzero q -corner γ from ϕ to ψ , so Θ below is q -positive:

$$\Theta = \begin{pmatrix} \phi & \gamma \\ \gamma^* & \psi \end{pmatrix}.$$

Note that

$$L_\Theta = \begin{pmatrix} \phi & \sigma \\ \sigma^* & L_\psi \end{pmatrix},$$

where $\sigma = \lim_{t \rightarrow \infty} t\gamma(I+t\gamma)^{-1}$ is a corner from ϕ to L_ψ since L_Θ is completely positive. Furthermore, $\sigma^2 = \sigma$, $\gamma = \sigma \circ \gamma = \gamma \circ \sigma$, $\text{range}(\sigma) = \text{range}(\gamma)$, and $\text{nullspace}(\sigma) = \text{nullspace}(\gamma)$.

Of course, ϕ has the form $\phi(A) = \rho(A)I$ for some state $\rho \in M_n(\mathbb{C})^*$. Suppose that ρ is faithful. Let $A \in M_{n \times k}(\mathbb{C})$ be any norm one matrix in the range of σ , and let $P \in M_n(\mathbb{C})$ be the orthogonal projection onto $\text{range}(A) \subseteq \mathbb{C}^n$, so $PA = A$ and $A^*P = A^*$. Applying L_Θ to the positive matrix $Q \in M_{n+k}(\mathbb{C})$ given by

$$Q = \begin{pmatrix} P & 0 \\ 0 & I_k \end{pmatrix} \begin{pmatrix} I_n & A \\ A^* & I_k \end{pmatrix} \begin{pmatrix} P & 0 \\ 0 & I_k \end{pmatrix} = \begin{pmatrix} P & PA \\ A^*P & I_k \end{pmatrix} = \begin{pmatrix} P & A \\ A^* & I_k \end{pmatrix},$$

we see from complete positivity of L_Θ that

$$L_\Theta(Q) = \begin{pmatrix} \rho(P)I & A \\ A^* & I_k \end{pmatrix} \geq 0.$$

Since $\|A\| = 1$, positivity of the above matrix implies that $\rho(P) = 1$, hence $P = I_n$ by faithfulness of ρ . Since P is the orthogonal projection onto the range of A , we have $\text{rank}(A) = n$. We conclude that every nonzero element of $\text{range}(\sigma)$ has rank n .

For some matrix unit $e_{ij} \in M_{n \times k}(\mathbb{C})$, we have $M := \sigma(e_{ij}) \neq 0$, so $\text{rank}(M) = n$. By complete positivity of L_Θ , the matrix R below must be positive:

$$R = L_\Theta \begin{pmatrix} e_{ii} & e_{ij} \\ e_{ji} & e_{jj} \end{pmatrix} = \begin{pmatrix} \rho(e_{ii})I_n & M \\ M^* & e_{jj} \end{pmatrix}.$$

However, R is not positive. Indeed, since $\text{rank}(M) = n \geq 2$, there exists a vector $g \in \mathbb{C}^k$ such that $e_{jj}g = 0$ but $Mg \neq 0$. For all $\lambda \in \mathbb{R}$, we have

$$\left\langle \begin{pmatrix} Mg \\ -\lambda g \end{pmatrix}, R \begin{pmatrix} Mg \\ -\lambda g \end{pmatrix} \right\rangle = (\rho(e_{ii}) - 2\lambda) \|Mg\|^2,$$

which is negative whenever $\lambda > 1$. We conclude $R \not\geq 0$, contradicting complete positivity of L_Θ . Therefore, there is no nonzero q -corner from ϕ to ψ , so α^d and β^d are non-cocycle conjugate by Proposition 2.10.

Now suppose that ρ is not faithful, so for some mutually orthogonal norm one vectors $\{f_i\}_{i=1}^{p < n} \subset \mathbb{C}^n$ and positive numbers $\lambda_1, \dots, \lambda_p$ with $\sum_{i=1}^p \lambda_i = 1$, we have $\rho(A) = \sum_{i=1}^p \lambda_i (f_i, A f_i)$ for all $A \in M_n(\mathbb{C})$. For some unitary $U \in M_n(\mathbb{C})$ we have

$$\phi_U(A) = \rho(UAU^*)I = \left(\sum_{i=n-p+1}^n \lambda_{i-n+p} a_{ii} \right) I.$$

Since the E_0 -semigroup α_U^d induced by (ϕ_U, ν) is cocycle conjugate to α^d by Proposition 2.11, the theorem follows if we show that α_U^d is not cocycle conjugate to β^d .

If there is a hyper maximal q -corner γ from ϕ_U to ψ , then

$$\Theta = \begin{pmatrix} \phi_U & \gamma \\ \gamma^* & \psi \end{pmatrix} \geq_q 0,$$

and we have

$$L_\Theta = \begin{pmatrix} \phi_U & \sigma \\ \sigma^* & L_\psi \end{pmatrix},$$

where $\sigma = \lim_{t \rightarrow \infty} t\gamma(I + t\gamma)^{-1}$ is a norm one corner from ϕ_U to L_ψ such that $\sigma^2 = \sigma$ and $\text{range}(\sigma) = \text{range}(\gamma)$.

Note that $\phi_U(e_{11}) = 0$, hence $L_\Theta(e_{11}) = 0$, so by Lemma 3.5 we have

$$\sigma \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1k} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \equiv 0.$$

Therefore, for some $\ell : M_{(n-1) \times k}(\mathbb{C}) \rightarrow M_{1 \times k}(\mathbb{C})$ and $\tilde{\sigma} : M_{(n-1) \times k}(\mathbb{C}) \rightarrow M_{(n-1) \times k}(\mathbb{C})$, we have

$$\sigma \begin{pmatrix} B_{1 \times k} \\ A_{(n-1) \times k} \end{pmatrix} = \begin{pmatrix} \ell(A_{(n-1) \times k}) \\ \tilde{\sigma}(A_{(n-1) \times k}) \end{pmatrix}.$$

Note that since $\sigma^2 = \sigma$ and $\|\sigma\| = 1$, we have $\tilde{\sigma}^2 = \tilde{\sigma}$ and $\|\tilde{\sigma}\| = 1$.

We claim that $\ell \equiv 0$. To show this, we let $2 \leq i \leq n$ and $1 \leq j \leq k$ be arbitrary. Since L_Θ is completely positive, we have

$$0 \leq R := L_\Theta \begin{pmatrix} e_{ii} & e_{ij} \\ e_{ji} & e_{jj} \end{pmatrix} = \begin{pmatrix} \rho(e_{ii})I_n & \sigma(e_{ij}) \\ \sigma^*(e_{ji}) & e_{jj} \end{pmatrix}.$$

A rank argument similar to the one from the faithful case shows that $\text{rank}(\sigma(e_{ij})) \leq 1$ since R is positive. If $\sigma(e_{ij}) = 0$, then $\ell(e_{ij}) = 0$. If $\text{rank}(\sigma(e_{ij})) = 1$, we see from the form of R that $\sigma(e_{ij})$ is a column matrix of the form below for some scalars c_1, \dots, c_n :

$$\sigma(e_{ij}) = \sum_{m=1}^n c_m e_{mj}.$$

Since $\sigma^2 = \sigma$ and $\sigma(e_{1j}) = 0$, we have

$$(29) \quad \sum_{m=1}^n c_m e_{mj} = \sigma \left(\sum_{m=1}^n c_m e_{mj} \right) = \sigma \left(\sum_{m=2}^n c_m e_{mj} \right).$$

If $c_1 \neq 0$, then by equation (29),

$$\left\| \sigma \left(\sum_{m=2}^n c_m e_{mj} \right) \right\| = \left\| \sum_{m=1}^n c_m e_{mj} \right\| > \left\| \sum_{m=2}^n c_m e_{mj} \right\|,$$

contradicting the fact that σ is a contraction. Hence $c_1 = 0$, that is, $\ell(e_{ij}) = 0$. Since i and j were chosen arbitrarily, we conclude $\ell \equiv 0$. This means that

$$(30) \quad \text{range}(\sigma) \cap \text{span}\{e_{11}, e_{12}, \dots, e_{1n}\} = \{0\}.$$

The same holds for γ since $\text{range}(\gamma) = \text{range}(\sigma)$. Define $\Theta' : M_{n+k}(\mathbb{C}) \rightarrow M_{n+k}(\mathbb{C})$ by

$$\Theta'(A) = (I_{n+k} - e_{11})\Theta(A)(I_{n+k} - e_{11}).$$

From equation (30), we have

$$\Theta' = \begin{pmatrix} \phi' & \gamma \\ \gamma^* & \psi \end{pmatrix},$$

where $\phi' : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$ is the map $\phi'(A) = \rho(A)(I_n - e_{11})$. Note $(\phi')^2 = \phi'$ and ϕ' is q -positive, so $I + t\Theta'$ is invertible for all $t \geq 0$ and

$$\Theta'(I + t\Theta')^{-1}(A) = (I_{n+k} - e_{11}) \left[\Theta(I + t\Theta)^{-1}(A) \right] (I_{n+k} - e_{11})$$

for all $A \in M_{n+k}(\mathbb{C})$. This shows that $\Theta' \geq_q 0$. Furthermore, $\Theta \geq_q \Theta'$ since

$$\Theta'(I + t\Theta')^{-1} - \Theta(I + t\Theta)^{-1}(A) = e_{11}\Theta(I + t\Theta)^{-1}(A)e_{11}$$

for all $t \geq 0$, $A \in M_{n+k}(\mathbb{C})$. Trivially, $\phi' \neq \phi$, contradicting hyper maximality of γ . We conclude there is no hyper maximal q -corner from ϕ_U to ψ , hence α^d and β^d are non-cocycle conjugate by Proposition 2.10. □

We conclude with the following:

Corollary 5.2. *Let $\phi_1 : M_2(\mathbb{C}) \rightarrow M_2(\mathbb{C})$ be a unital rank one q -positive map. Let $\phi_2 : M_2(\mathbb{C}) \rightarrow M_2(\mathbb{C})$ be the diagonal map, and let $\phi_3 : M_2(\mathbb{C}) \rightarrow M_2(\mathbb{C})$ be a unital invertible q -positive Schur map. Suppose ν is a type II Powers weight of the form $\nu(\sqrt{I - \Lambda(1)}B\sqrt{I - \Lambda(1)}) = (f, Bf)$.*

The boundary weight doubles (ϕ_i, ν) and (ϕ_j, ν) induce cocycle conjugate E_0 -semigroups if and only if $i = j$.

Proof: For each i , let α_i^d be the E_0 -semigroup induced by (ϕ_i, ν) . Theorem 5.1 implies that α_1^d is not cocycle conjugate to α_2^d or α_3^d . A result at the end of [11] shows that α_2^d and α_3^d are non-cocycle conjugate, but we present a proof here for the sake of completeness. Let γ be any q -corner from ϕ_2 to ϕ_3 , so $\Theta : M_4(\mathbb{C}) \rightarrow M_4(\mathbb{C})$ below is q -positive:

$$\Theta = \begin{pmatrix} \phi_2 & \gamma \\ \gamma^* & \phi_3 \end{pmatrix}.$$

Applying Θ to the matrices $e_{11} + e_{1j} + e_{j1} + e_{jj}$ and $e_{22} + e_{2k} + e_{k2} + e_{kk}$ for $j, k = 3, 4$, we conclude from completely positivity of Θ that γ is a Schur map.

Form L_Θ , observing that

$$L_\Theta = \begin{pmatrix} \phi_2 & \sigma \\ \sigma^* & Id_{2 \times 2} \end{pmatrix},$$

where $\sigma = \lim_{t \rightarrow \infty} t\gamma(I + t\gamma)^{-1}$, $\sigma^2 = \sigma$, $\text{range}(\sigma) = \text{range}(\gamma)$, and $\text{nullspace}(\sigma) = \text{nullspace}(\gamma)$. Note that σ is also a Schur map, and write σ in the form

$$\sigma(A) = \begin{pmatrix} z_{11}a_{11} & z_{12}a_{12} \\ z_{21}a_{21} & z_{22}a_{22} \end{pmatrix}.$$

Since $\sigma^2 = \sigma$, we have $z_{ij} \in \{0, 1\}$ for each i and j .

We claim that

$$(31) \quad z_{21} = z_{22} = 1 \quad \text{or} \quad z_{21} = z_{22} = 0.$$

To prove this, first suppose that $z_{21} = 1$. Let $T \in M_4(\mathbb{C})$ be the positive matrix whose entries are all 1. Then

$$0 \leq L_\Theta\left((I - e_{11})T(I - e_{11})\right) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & z_{22} \\ 0 & 1 & 1 & 1 \\ 0 & z_{22} & 1 & 1 \end{pmatrix}.$$

Since the above matrix is positive, the determinant of its bottom right 3×3 minor must be nonnegative. This quantity is $-(z_{22} - 1)^2$, hence $z_{22} = 1$. On the other hand, if $z_{21} = 0$, then

$$0 \leq \det \left[L_\Theta\left((I - e_{11})T(I - e_{11})\right) \right] = -(z_{22})^2,$$

so $z_{22} = 0$, yielding (31).

Analogous observations regarding $L_\Theta\left((I - e_{22})T(I - e_{22})\right)$ show that

$$(32) \quad z_{11} = z_{12} = 1 \quad \text{or} \quad z_{11} = z_{12} = 0.$$

By equations (31) and (32), σ is the Schur mapping $\sigma(A) = M_j \bullet A$ for one of the three matrices below:

$$M_1 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \quad \text{or} \quad M_3 = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}.$$

Note that $\sigma(A) = M_1 \bullet A$ is not a corner from ϕ_2 to $Id_{2 \times 2}$, since in that case we would have

$$L_\Theta(T) = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \not\geq 0,$$

contradicting complete positivity of L_Θ .

Recall that γ is a Schur map and $\text{range}(\sigma) = \text{range}(\gamma)$. Therefore, if $\sigma(A) = M_2 \bullet A$ for all $A \in M_2(\mathbb{C})$, then $\gamma(A) = R \bullet A$ for some matrix R such that $r_{21} = r_{22} = 0$. Letting $S = e_{11} + e_{33} + e_{44}$ and defining Θ' by

$$\Theta'(A) = S\Theta(A)S$$

for all $A \in M_4(\mathbb{C})$, we see that Θ' is completely positive by construction and

$$\Theta' = \begin{pmatrix} \phi'_2 & \gamma \\ \gamma^* & \phi_3 \end{pmatrix}$$

where $\phi'_2(A) = a_{11}e_{11}$ for all $A \in M_2(\mathbb{C})$. Furthermore, we have

$$\Theta'(I + t\Theta')^{-1}(A) = S\left(\Theta(I + t\Theta)^{-1}(A)\right)S$$

for all $t > 0$ and $A \in M_4(\mathbb{C})$, so Θ' is q -positive. Also, $\Theta \geq_q \Theta'$ since

$$\Theta(I + t\Theta)^{-1}(A) - \Theta'(I + t\Theta')^{-1}(A) = \frac{1}{1+t}e_{22}Ae_{22}$$

for all $A \in M_4(\mathbb{C})$ and $t \geq 0$. Therefore, γ is not a hyper maximal q -corner.

If $\sigma(A) = M_3 \bullet A$, then we argue precisely as we just did, noting first that γ is a Schur map $\gamma(A) = Y \bullet A$ for some $Y \in M_2(\mathbb{C})$ with $y_{11} = y_{12} = 0$. Letting $X = e_{22} + e_{33} + e_{44}$ and defining ϕ_2'' by $\phi_2''(A) = a_{22}e_{22}$, we note that the map $\Theta'' : M_4(\mathbb{C}) \rightarrow M_4(\mathbb{C})$ defined by

$$\Theta'' = \begin{pmatrix} \phi_2'' & \gamma \\ \gamma^* & \phi_3 \end{pmatrix}$$

satisfies $\Theta \neq \Theta''$ and $\Theta \geq_q \Theta''$. We conclude that γ is not a hyper maximal q -corner.

We have shown that no q -corner γ from ϕ_2 and ϕ_3 is hyper maximal, hence α_2^d and α_3^d are non-cocycle conjugate by Proposition 2.10. □

Acknowledgments

The author would like to thank Robert Powers for his guidance and enthusiastic interest in this research.

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DEPARTMENT OF MATHEMATICS
BEN-GURION UNIVERSITY OF THE NEGEV
P.O. Box 653
BE'ER SHEVA 84105, ISRAEL
E-mail address: `cjankows@math.bgu.ac.il`